The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all.

**Exercise 1.**

Let us define the \( \alpha \)-sub-level set of a function \( f : \mathbb{R}^n \to \mathbb{R} \) to be the set \( S_\alpha \) defined as \( \{ x : f(x) \leq \alpha \} \).

(i) Prove that if a function \( f \) is convex, then all its sub-level sets are convex sets.

(ii) Is it true that a function whose sub-level sets are all convex is necessarily convex?

**Exercise 2.**

Recall the definition of Laplacian \( L = BR^{-1}B^\top \).

(i) We can also define Laplacian as \( L \) defined as \( D - A \), where \( A \) is the weighted adjacency matrix, i.e.\( A(u, v) = 1/r(u, v) \), and \( D \) defined as \( \text{diag}_v \in V w(v) \) for \( w(v) := \sum_{(u,v) \in E} 1/r(u, v) \). Prove that these two definitions are equivalent.

(ii) Given a function on the vertices, \( x \in \mathbb{R}^V \), the Laplacian quadratic form is

\[
\sum_{(u,v) \in E} \frac{(x(u) - x(v))^2}{r(u,v)}.
\]

Prove the above equality and building on that, show that \( L \) is positive semi-definite.

(iii) What is the kernel of \( L \), which is denoted by \( \text{Ker}(L) \)?

**Exercise 3.**

(i) Prove that for a matrix \( A \) we have \( \text{im}(A) = \ker(A^\top)^\perp \), where \( \text{im}(A) \) denotes the image of \( A \) and \( \ker(A^\top)^\perp \) is the orthogonal complement to \( \ker(A^\top) \).

(ii) Building on part (i), prove that in our flow problem, when the graph is connected, an electrical flow \( f \) routing \( d \) exists if and only if \( 1^\top d = 0 \).

**Exercise 4.**

Define the gradient of a multivariate function \( f : S \to \mathbb{R} \) for \( S \subseteq \mathbb{R}^n \). Then, prove that the system of linear equations \( Lx = d \) is the same as the system obtained by setting the gradient with respect to \( x \) of the function \( c(x) = \frac{1}{2}x^\top Lx - x^\top d \) equal to zero.
Exercise 5.

(i) Recall that the electrical flow and voltages satisfy $f^* = R^{-1}B^\top x^*$ and $Bf^* = d$. Prove that $(f^*)^\top Rf^* = (x^*)^\top Lx^*$.

(ii) Conclude that

$$
\max_{x \in \mathbb{R}^V} x^\top d - \frac{1}{2} x^\top L x = \min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r(e)f(e)^2
$$

s.t. $Bf = d$. 