The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all.

Exercise 1.

Prove that

1. $v_0 = x_0 - \frac{a_0}{\sigma} \nabla f(x_0)$
2. $L_0 = f(x_0) - \frac{a_0}{2\sigma} \|\nabla f(x_0)\|_2^2 - \frac{\sigma}{2a_0} \|x^* - x_0\|_2^2.$

Exercise 2.

Prove that

1. $m_i(v) = m_i(v_i) + \frac{\sigma}{2} \|v - v_i\|_2^2$
2. $m_{i+1}(v) = m_i(v) + a_{i+1} f(x_{i+1}) + (a_{i+1} \nabla f(x_{i+1}), v - x_{i+1})$
3. $v_{i+1} = v_i - \frac{a_0}{\sigma} \nabla f(x_0)$.

Exercise 3.

1. Assume that $S \subseteq \mathbb{R}^n$ is a convex set and that the function $f : S \to \mathbb{R}$ is convex. Suppose that $x_1, \ldots, x_n \in S$ and $\theta_1, \ldots, \theta_n \geq 0$ with $\theta_1 + \cdots + \theta_n = 1$. Prove that

$$f(\theta_1 x_1 + \cdots + \theta_n x_n) \leq \theta_1 f(x_1) + \cdots + \theta_n f(x_n).$$

**Remark.** This is typically known as Jensen’s inequality and can be extended to infinite sums. If $\mathcal{D}$ is a probability distribution on $S$, and $X \sim \mathcal{D}$, then

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$$

whenever both integrals are finite.

2. Prove that

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} x_i.$$

3. Prove that

$$\frac{1}{\frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i}} \leq \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.$$