Advanced Graph Algorithms and Optimization Spring 2024

Spectral Graph Theory

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These exercises will not count toward your grade, but you are encouraged to solve them all. This exercise sheet contains exercises relating to lectures in Weeks 3. We encourage you to start the exercises early so you have time to get through everything.

To get feedback, you must hand in your solutions by 23:59 on March 14th. Both hand-written and LaTeX solutions are acceptable, but we will only attempt to read legible text.

Exercise 1

Let P_n be the path from vertex 1 to n and $G_{1,n}$ be the graph with only the edge between vertex 1 and n. Furthermore, assume that the edge between vertex i and $i+1$ has positive weight w_i for $1 \leq i \leq n-1$. Prove that

$$G_{1,n} \preceq \left(\sum_{i=1}^{n-1} \frac{1}{w_i} \right) \sum_{i=1}^{n-1} w_i G_{i,i+1}.$$

Exercise 2

In Chapter 4, we proved that

$$\lambda_2(T_n) \geq \frac{1}{2n \log_2 n} \, .$$

Improve this bound to $\lambda_2(T_n) \geq 1/cn$ for some constant $c > 0$. You may assume $n = 2^{d+1} - 1$ for some integer d.

Hint: Use the result of previous exercise.

Exercise 3

Find the conductance $\phi \in (0, 1]$ for the following graphs:

1. the complete graph K_n over n vertices.
2. the path graph P_n over n vertices.

Exercise 4

Show that $\lambda_2(\boldsymbol{L}) \neq 0$ if and only if G is connected. Argue that the same applies for $\boldsymbol{N}$.

Exercise 5

A quite related concept to conductance is *sparsity*: we define the sparsity of a cut $\emptyset \subset S \subset V$ by

$$\sigma(S) = \frac{|E(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}}.$$

An alternative version of Cheeger's inequality relates the second eigenvalue of $\boldsymbol{L}$ (not $\boldsymbol{N}$) to the sparsity of the graph $\sigma(G) = \min_{\emptyset \subset S \subset V} \sigma(S)$:

$$\frac{\lambda_2(\boldsymbol{L})}{2} \leq \sigma(G) \leq \sqrt{2d_{max} \cdot \lambda_2(\boldsymbol{L})}$$

where d_{max} is the maximum degree of any vertex in the graph.

Prove the lower bound on $\sigma(G)$, i.e. that $\frac{\lambda_2(\boldsymbol{L})}{2} \leq \sigma(G)$.

Hint: Follow closely the proof of of the lower bound in Cheeger's inequality and try to understand what has to be adapted.

Exercise 6

In the lecture, we skipped various steps in the proof of Cheeger's inequality. Show that

1. $\boldsymbol{N}$ is symmetric and in fact PSD. We recommend to prove this by proving the following stronger statement: for any matrix $\boldsymbol{A}$ that is PSD, and any matrix $\boldsymbol{C}$, we have that $\boldsymbol{C}^\top \boldsymbol{A} \boldsymbol{C}$ is PSD.

2. Show that the normalization of $\boldsymbol{z}$ in the upper bound proof of Cheeger's inequality can only make the ratio we are interested in smaller. I.e. prove that $\frac{\boldsymbol{z}^\top \boldsymbol{L} \boldsymbol{z}}{\boldsymbol{z}^\top \boldsymbol{D} \boldsymbol{z}} \geq \frac{\boldsymbol{z}_{sc}^\top \boldsymbol{L} \boldsymbol{z}_{sc}}{\boldsymbol{z}_{sc}^\top \boldsymbol{D} \boldsymbol{z}_{sc}}$.

   Hint: Argue first about the transformation of $\boldsymbol{z}$ to $\boldsymbol{z}_c$. One way of relating their denominator is by minimizing over all choices of $\boldsymbol{z}_c$ for c. For $\boldsymbol{z}_c$ and $\boldsymbol{z}_{sc}$ you should be able to prove an equality.

3. We have also skipped proving that τ is sampled according to a valid probability distribution: Show that $\int_\tau \mathbb{P}[\tau = \ell] \, d\tau = 1$.

   Hint: Recall the properties of $\boldsymbol{z}_{sc}$.

4. Show that

   $$\mathbb{E}_\tau[|E(S_\tau, V \setminus S_\tau)|] \leq \sum_{\{i,j\} \in E} |\boldsymbol{z}_{sc}(i) - \boldsymbol{z}_{sc}(j)| \cdot (|\boldsymbol{z}_{sc}(i)| + |\boldsymbol{z}_{sc}(j)|)$$

   by concluding the argument in the proof.

5. Standard Probabilistic Method: Consider a random variable X with a discrete distribution and let Ω be the sample space. Argue that there exists an $\omega \in \Omega$ with $X(\omega) \geq \mathbb{E}[X]$.

   Hint: Recall the definition of expectation of a discrete random variable.

6. Using the probabilisitic method for Cheeger's Inequality: recall that in our proof, we want to argue that $\frac{\mathbb{E}_\tau\left[\mathbf{1}_S^\top \boldsymbol{L}\mathbf{1}_S\right]}{\mathbb{E}_\tau\left[\mathbf{1}_S^\top \boldsymbol{D}\mathbf{1}_S\right]} \leq \sqrt{2 \cdot \frac{\boldsymbol{z}_{sc}^\top \boldsymbol{L}\boldsymbol{z}_{sc}}{\boldsymbol{z}_{sc}^\top \boldsymbol{D}\boldsymbol{z}_{sc}}}$ implies that there exists an S with $\frac{\mathbf{1}_S^\top \boldsymbol{L}\mathbf{1}_S}{\mathbf{1}_S^\top \boldsymbol{D}\mathbf{1}_S} \leq \sqrt{2 \cdot \frac{\boldsymbol{z}_{sc}^\top \boldsymbol{L}\boldsymbol{z}_{sc}}{\boldsymbol{z}_{sc}^\top \boldsymbol{D}\boldsymbol{z}_{sc}}}$. There are two ways to prove this (feel free to choose just one):

   (a) you can prove this claim by considering $\mathbb{E}_\tau\left[\mathbf{1}_S^\top \boldsymbol{L}\mathbf{1}_S\right] \leq \sqrt{2 \cdot \frac{\boldsymbol{z}_{sc}^\top \boldsymbol{L}\boldsymbol{z}_{sc}}{\boldsymbol{z}_{sc}^\top \boldsymbol{D}\boldsymbol{z}_{sc}}} \cdot \mathbb{E}_\tau\left[\mathbf{1}_S^\top \boldsymbol{D}\mathbf{1}_S\right]$. Use only linearity of expectation to obtain an expression with a single $\mathbb{E}_\tau$ and apply the probabilistic method, or

   (b) you can prove that for any two discrete random variables $X, Y > 0$ with the same distribution, we have that there exists an $\omega \in \Omega$ with

$$\frac{X(\omega)}{Y(\omega)} \leq \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}.$$