Advanced Graph Algorithms and Optimization	Spring 2024

Spectral Graph Theory

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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all. This exercise sheet has exercises related to week 5. We encourage you to start early so you have time to go through everything.

To get feedback, you must hand in your solutions by 23:59 on March 28. Both hand-written and LaTeX solutions are acceptable, but we will only attempt to read legible text.

Notation

Througout the following exercises, we will use the following notation:

- S^n is the set of symmetric real matrices $n \times n$ matrices.
- S^n_+ is the set of positive semi-definite $n \times n$ matrices.
- S^n_{++} is the set of positive definite $n \times n$ matrices.

Whenever we say a matrix is positive semi-definite or positive definite, we require it to be real and symmetric.

Exercise 1

1. Show that there exist two matrices $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$ such that $\boldsymbol{A} \preceq \boldsymbol{B}$ but $\boldsymbol{A}^2 \npreceq \boldsymbol{B}^2$.
2. Let $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$, and assume $\boldsymbol{A} \preceq \boldsymbol{B}$. Prove that $\boldsymbol{B}^{-1} \preceq \boldsymbol{A}^{-1}$.
   Hint: It might help to first prove that for a matrix $\boldsymbol{C} \in \mathbb{R}^{n \times n}$, we have $\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^\top \preceq \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^\top$.

Exercise 2

Suppose that a weighted graph G is a ϕ-expander, with Laplacian $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$.

1. Prove that for any $\boldsymbol{z} \perp \boldsymbol{1}$,
   $$\boldsymbol{z}^\top \boldsymbol{L}^\dagger \boldsymbol{z} \leq 2\phi^{-2} \boldsymbol{z}^\top \boldsymbol{D}^{-1} \boldsymbol{z}.$$
   Hint: Use the result from Exercise 5 in Problem Set 4.
2. Use the statement above to give an upper bound on the effective resistance between any two vertices u, v of G.

Exercise 3

In this exercise, we want you to complete the proof of Theorem 8.3.3 in Chapter 8. Refer to the lectures notes for definitions of the terms used here.

1. Prove that Equation (8.4) is satisfied, i.e. that for all edges $e \in E$ we have $\|\boldsymbol{X}_e\| \leq \frac{1}{\alpha}$.
2. Prove that Equation (8.5) is satisfied, i.e. that $\left\|\sum_e \mathbb{E}\left[\boldsymbol{X}_e^2\right]\right\| \leq \frac{1}{\alpha}$.
3. Explain how we can use a scalar Chernoff bound to prove that $\left|\tilde{E}\right| \leq O(\epsilon^{-2}\log(n/\delta)n)$ with probability at least $1-\delta/2$. You may pick any constant that suits you to establish the $O(\cdot)$ bound.