Advanced Graph Algorithms and Optimization	Spring 2024
Spectral Graph Theory	
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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all. This exercise sheet has exercises related to week 5. We encourage you to start early so you have time to go through everything.

To get feedback, you must hand in your solutions by 23:59 on March 28. Both hand-written and LaTeX solutions are acceptable, but we will only attempt to read legible text.

Notation

Througout the following exercises, we will use the following notation:

- S^n is the set of symmetric real matrices $n \times n$ matrices.
- S^n_+ is the set of positive semi-definite $n \times n$ matrices.
- S^n_{++} is the set of positive definite $n \times n$ matrices.

Whenever we say a matrix is positive semi-definite or positive definite, we require it to be real and symmetric.

Exercise 1

1. Show that there exist two matrices $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$ such that $\boldsymbol{A} \preceq \boldsymbol{B}$ but $\boldsymbol{A}^2 \not\preceq \boldsymbol{B}^2$.
2. Let $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$, and assume $\boldsymbol{A} \preceq \boldsymbol{B}$. Prove that $\boldsymbol{B}^{-1} \preceq \boldsymbol{A}^{-1}$.
   Hint: It might help to first prove that for a matrix $\boldsymbol{C} \in \mathbb{R}^{n\times n}$, we have $\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^\top \preceq \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^\top$.

Solution

1. Suppose that
   $$\boldsymbol{A} := \begin{bmatrix} 26 & 5 \\ 5 & 2 \end{bmatrix}$$
   and
   $$\boldsymbol{B} := \begin{bmatrix} 51 & 0 \\ 0 & 3 \end{bmatrix}.$$
   Consider an arbitrary vector $\boldsymbol{x} \in \mathbb{R}^2$. Then, we have that
   $$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = 26\boldsymbol{x}(1)^2 + 10\boldsymbol{x}(1)\boldsymbol{x}(2) + 2\boldsymbol{x}(2)^2.$$

and

$$\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} = 51\boldsymbol{x}(1)^2 + 3\boldsymbol{x}(2)^2.$$

Therefore, we get

$$\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} - \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = 25\boldsymbol{x}(1)^2 + \boldsymbol{x}(2)^2 - 10\boldsymbol{x}(1)\boldsymbol{x}(2) = (5\boldsymbol{x}(1) - \boldsymbol{x}(2))^2 \geq 0$$

which implies that $\boldsymbol{A} \preceq \boldsymbol{B}$.

Furthermore, $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$ because for any $\boldsymbol{x} \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$,

$$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = 26\boldsymbol{x}(1)^2 + 10\boldsymbol{x}(1)\boldsymbol{x}(2) + 2\boldsymbol{x}(2)^2 = (5\boldsymbol{x}(1) + \boldsymbol{x}(2))^2 + \boldsymbol{x}(1)^2 + \boldsymbol{x}(2)^2 > 0.$$

On the other hand, for $\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have $\boldsymbol{x}^\top \boldsymbol{A}^2\boldsymbol{x} = 29$ and $\boldsymbol{x}^\top \boldsymbol{B}^2\boldsymbol{x} = 9$ which implies that $\boldsymbol{A}^2 \not\preceq \boldsymbol{B}^2$.

2. Let $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$, and assume that $\boldsymbol{A} \preceq \boldsymbol{B}$. Let us first prove that for any $\boldsymbol{C} \in \mathbb{R}^{n\times n}$,

$$\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^\top \preceq \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^\top. \tag{1}$$

For $\boldsymbol{x} \in \mathbb{R}^n$, we let $\boldsymbol{y} = \boldsymbol{C}^\top\boldsymbol{x}$. Then, we get

$$\boldsymbol{x}^\top \left(\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^\top - \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^\top\right)\boldsymbol{x} = \left(\boldsymbol{C}^\top\boldsymbol{x}\right)^\top (\boldsymbol{A} - \boldsymbol{B})\left(\boldsymbol{C}^\top\boldsymbol{x}\right) = \boldsymbol{y}^\top (\boldsymbol{A} - \boldsymbol{B})\,\boldsymbol{y} \leq 0.$$

Now, we prove that $\boldsymbol{B}^{-1} \preceq \boldsymbol{A}^{-1}$. We know that $\boldsymbol{B}^{-\frac{1}{2}}$ exists and $\boldsymbol{0} \prec \boldsymbol{B}^{-\frac{1}{2}}$ since $\boldsymbol{0} \prec \boldsymbol{B}$. By applying Equation (1), we have

$$\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{B}^{-\frac{1}{2}} \preceq \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{B}^{-\frac{1}{2}} = \boldsymbol{I}.$$

Furthermore, since $\boldsymbol{0} \prec \boldsymbol{A}$, we get

$$\left(\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{B}^{-\frac{1}{2}}\right)^{-1} = \boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}} \succeq \boldsymbol{I}.$$

By applying Equation (1) another time, we conclude that

$$\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}} \succeq \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{I}\boldsymbol{B}^{-\frac{1}{2}} \Rightarrow \boldsymbol{A}^{-1} \succeq \boldsymbol{B}^{-1}.$$

Exercise 2

Suppose that a weighted graph G is a ϕ-expander, with Laplacian $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$.

1. Prove that for any $\boldsymbol{z} \perp \boldsymbol{1}$,

$$\boldsymbol{z}^\top \boldsymbol{L}^\dagger \boldsymbol{z} \leq 2\phi^{-2}\boldsymbol{z}^\top \boldsymbol{D}^{-1}\boldsymbol{z}.$$

Hint: Use the result from Exercise 5 in Problem Set 4.

2. Use the statement above to give an upper bound on the effective resistance between any two vertices u, v of G.

Solution.

1. Note that the null space of $\boldsymbol{D}^{-1/2}\boldsymbol{L}\boldsymbol{D}^{-1/2}$ is spanned by $\boldsymbol{D}^{1/2}\mathbf{1}$, as $\mathbf{1}$ spans the null space of $\boldsymbol{L}$. Cheeger's inequality gives that for $\boldsymbol{y}\perp\boldsymbol{D}^{1/2}\mathbf{1}$,

$$\boldsymbol{y}^\top \boldsymbol{D}^{-1/2}\boldsymbol{L}\boldsymbol{D}^{-1/2}\boldsymbol{y} \geq 0.5\phi^2 \boldsymbol{y}^\top \boldsymbol{y}.$$

We let $\mathbf{Q}$ denote the projection orthogonal to $\boldsymbol{D}^{1/2}\mathbf{1}$. We can then equivalently write

$$\boldsymbol{D}^{-1/2}\boldsymbol{L}\boldsymbol{D}^{-1/2} \succeq 0.5\phi^2\mathbf{Q}.$$

From this we conclude that,

$$(\boldsymbol{D}^{-1/2}\boldsymbol{L}\boldsymbol{D}^{-1/2})^\dagger \preceq 2\phi^{-2}\mathbf{Q}^\dagger$$

as $\boldsymbol{A} \succeq \boldsymbol{B}$ implies $\boldsymbol{A}^\dagger \preceq \boldsymbol{B}^\dagger$ when $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same null space. Hence by using Exercise 12 from problem set 2 and $\mathbf{Q} = \mathbf{Q}^\dagger$, we then get

$$\mathbf{Q}\boldsymbol{D}^{1/2}\boldsymbol{L}^\dagger\boldsymbol{D}^{1/2}\mathbf{Q} \preceq 2\phi^{-2}\mathbf{Q}.$$

This we can rewrite as for all $\boldsymbol{y}\perp\boldsymbol{D}^{1/2}\mathbf{1}$.

$$\boldsymbol{y}^\top\boldsymbol{D}^{1/2}\boldsymbol{L}^\dagger\boldsymbol{D}^{1/2}\boldsymbol{y} \leq 2\phi^{-2}\boldsymbol{y}^\top\boldsymbol{y}.$$

Substituting $\boldsymbol{z} = \boldsymbol{D}^{1/2}\boldsymbol{y}$ changes the constraint to $\boldsymbol{D}^{-1/2}\boldsymbol{z}\perp\boldsymbol{D}^{1/2}\mathbf{1}$ i.e. $\boldsymbol{z}\perp\mathbf{1}$. Thus we have that for all $\boldsymbol{z}\perp\mathbf{1}$.

$$\boldsymbol{z}^\top\boldsymbol{L}^\dagger\boldsymbol{z} \preceq 2\phi^{-2}\boldsymbol{z}^\top\boldsymbol{D}^{-1}\boldsymbol{z}.$$

2. If $\boldsymbol{z} = \boldsymbol{e}_u - \boldsymbol{e}_v$, then $\boldsymbol{z} \perp \mathbf{1}$, so

$$\boldsymbol{z}^\top\boldsymbol{L}^\dagger\boldsymbol{z} \leq 2\phi^{-2}\boldsymbol{z}^\top\boldsymbol{D}^{-1}\boldsymbol{z} = 2\phi^{-2}\left(\frac{1}{\boldsymbol{d}(u)} + \frac{1}{\boldsymbol{d}(v)}\right)$$

Exercise 3

In this exercise, we want you to complete the proof of Theorem 8.3.3 in Chapter 8. Refer to the lectures notes for definitions of the terms used here.

1. Prove that Equation (8.4) is satisfied, i.e. that for all edges $e \in E$ we have $\|\boldsymbol{X}_e\| \leq \frac{1}{\alpha}$.
2. Prove that Equation (8.5) is satisfied, i.e. that $\left\|\sum_e \mathbb{E}\left[\boldsymbol{X}_e^2\right]\right\| \leq \frac{1}{\alpha}$.
3. Explain how we can use a scalar Chernoff bound to prove that $\left|\tilde{E}\right| \leq O(\epsilon^{-2}\log(n/\delta)n)$ with probability at least $1-\delta/2$. You may pick any constant that suits you to establish the $O(\cdot)$ bound.

Solution.

1. We recall that

$$\boldsymbol{Y}_e = \begin{cases} \frac{\boldsymbol{w}(e)}{p_e}\boldsymbol{b}_e\boldsymbol{b}_e^\top & \text{with probability } p_e \\ \boldsymbol{0} & \text{otherwise} \end{cases}$$

for $p_e = \min(1, \alpha \left\| \Phi(\boldsymbol{w}(e)\boldsymbol{b}_e\boldsymbol{b}_e^\top) \right\|)$ and $\Phi(\boldsymbol{A}) = \boldsymbol{L}^{+/2}\boldsymbol{A}\boldsymbol{L}^{+/2}$. Furthermore, we have $\boldsymbol{X}_e = \Phi(\boldsymbol{Y}_e) - \mathbb{E}[\Phi(\boldsymbol{Y}_e)]$. Thus,

$$\boldsymbol{X}_e = \begin{cases} (\frac{1}{p_e} - 1)\Phi(\boldsymbol{w}(e)\boldsymbol{b}_e\boldsymbol{b}_e^\top) & \text{with probability } p_e \\ -\Phi(\boldsymbol{w}(e)\boldsymbol{b}_e\boldsymbol{b}_e^\top) & \text{otherwise.} \end{cases}$$

Note that when $p_e = 1$, we have $\boldsymbol{X}_e = \boldsymbol{0}$ always. So we only need to bound the norm when $p_e = \alpha \left\| \Phi(\boldsymbol{w}(e)\boldsymbol{b}_e\boldsymbol{b}_e^\top) \right\|$. Therefore, we can compute

$$\|\boldsymbol{X}_e\| \le \max\{\frac{1}{p_e} - 1, 1\} \left\| \Phi(\boldsymbol{w}(e)\boldsymbol{b}_e\boldsymbol{b}_e^\top) \right\| \le \frac{1}{p_e} \left\| \Phi(\boldsymbol{w}(e)\boldsymbol{b}_e\boldsymbol{b}_e^\top) \right\| = \frac{1}{\alpha}.$$

2. Note that $\sum_e \mathbb{E}\left[\boldsymbol{X}_e^2\right] \succeq 0$, because each term is PSD. So we only need to give an upper bound to bound the norm. Let us upper-bound $\mathbb{E}\left[\boldsymbol{X}_e^2\right]$. We can focus on $p_e = \alpha \left\| \Phi(\boldsymbol{w}(e)\boldsymbol{b}_e\boldsymbol{b}_e^\top) \right\|$, because in the other case $\boldsymbol{X}_e$ is identically zero. We have that

$$\begin{aligned} \mathbb{E}\left[\boldsymbol{X}_e^2\right] &= \left(\frac{1}{p_e} - 1\right) \Phi\left(\boldsymbol{w}\left(e\right) \boldsymbol{b}_e\boldsymbol{b}_e^\top\right)^2 \\ &\preceq \left(\frac{1}{p_e} - 1\right) \left\| \Phi\left(\boldsymbol{w}(e)\boldsymbol{b}_e\boldsymbol{b}_e^\top\right) \right\| \Phi\left(\boldsymbol{w}(e)\boldsymbol{b}_e\boldsymbol{b}_e^\top\right) \\ &\preceq \frac{1}{\alpha}\Phi\left(\boldsymbol{w}(e)\boldsymbol{b}_e\boldsymbol{b}_e^\top\right). \end{aligned}$$

Using the fact that $\|\Phi(\boldsymbol{L})\| \le 1$, we can conclude

$$\left\| \sum_e \mathbb{E}\left[\boldsymbol{X}_e^2\right] \right\| \le \left\| \frac{1}{\alpha}\Phi(\boldsymbol{L}) \right\| \le \frac{1}{\alpha}.$$

3. For each edge e, let Bernoulli random variable Z_e be 1 with probability p_e. Then, $\left|\tilde{E}\right| = \sum_e Z_e$ is the sum of $|E|$ independent Bernoulli random variables. Let us define $\mu' = 40\epsilon^{-2}n\log(n/\delta)$. We know that $\mathbb{E}\left[\left|\tilde{E}\right|\right] \le \mu'$. Therefore, by applying the Chernoff bound we get

$$\begin{aligned} \Pr\left[\left|\tilde{E}\right| \ge 2\mu'\right] &\le Pr\left[\left|\tilde{E}\right| \ge 2\,\mathbb{E}\left[\left|\tilde{E}\right|\right]\right] \\ &\le \exp\left(-\frac{\mathbb{E}\left[\left|\tilde{E}\right|\right]}{4}\right) \\ &\le \exp\left(-\frac{c\epsilon^{-2}n\log(n/\delta)}{4}\right) \\ &= \left(\frac{\delta}{n}\right)^{\frac{c\epsilon^{-2}n}{4}} \\ &\le \frac{\delta}{2} \end{aligned}$$

where we used that we can lower bound $\mathbb{E}\left[\left|\tilde{E}\right|\right]$ by $c\epsilon^{-2} n \log(n/\delta)$ for some constant $c > 0$.