Advanced Graph Algorithms and Optimization Spring 2024

Spectral Graph Theory

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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all. This exercise sheet has exercises related to week 6. We encourage you to start early so you have time to go through everything.

To get feedback, you must hand in your solutions by 23:59 on April 18. Both hand-written and LaTeX solutions are acceptable, but we will only attempt to read legible text.

Notation

Througout the following exercises, we will use the following notation:

- S^n is the set of symmetric real $n \times n$ matrices.
- S^n_+ is the set of positive semi-definite $n \times n$ matrices.
- S^n_{++} is the set of positive definite $n \times n$ matrices.

Whenever we say a matrix is positive semi-definite or positive definite, we require it to be real and symmetric.

Exercise 1

Consider $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^n$.

1. Assume that $\boldsymbol{I} + \boldsymbol{u}\boldsymbol{v}^\top$ is invertible. Determine c such that
$$\left(\boldsymbol{I} + \boldsymbol{u}\boldsymbol{v}^\top\right)^{-1} = \boldsymbol{I} - \frac{\boldsymbol{u}\boldsymbol{v}^\top}{c}.$$
2. Assume that both $\boldsymbol{A}$ and $\boldsymbol{A} + \boldsymbol{u}\boldsymbol{v}^\top$ are invertible. Prove that
$$\left(\boldsymbol{A} + \boldsymbol{u}\boldsymbol{v}^\top\right)^{-1} = \boldsymbol{A}^{-1} - \frac{\boldsymbol{A}^{-1}\boldsymbol{u}\boldsymbol{v}^\top\boldsymbol{A}^{-1}}{1 + \boldsymbol{v}^\top\boldsymbol{A}^{-1}\boldsymbol{u}}.$$
Hint: You might use that $(\boldsymbol{B}\boldsymbol{C})^{-1} = \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}$ *for two invertible matrices* $\boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{n\times n}$.

Exercise 2

Consider a matrix function $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$. For $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n\times n}$, we define

$$Df(\boldsymbol{X})[\boldsymbol{Y}] = \frac{\partial}{\partial t}\Big|_{t=0} f(\boldsymbol{X} + t\boldsymbol{Y}).$$

Remark. *Note that if we think of $\boldsymbol{X}$ and $\boldsymbol{Y}$ each as a vector of numbers, then this is the (matrix-valued) directional derivative of f at $\boldsymbol{X}$ in the direction of $\boldsymbol{Y}$.*

Consider $f(\boldsymbol{X}) = \boldsymbol{X}^{-1}$ for an invertible matrix $\boldsymbol{X} \in \mathbb{R}^{n \times n}$. Prove that

$$Df(\boldsymbol{X})[\boldsymbol{Y}] = -\boldsymbol{X}^{-1}\boldsymbol{Y}\boldsymbol{X}^{-1}.$$

Hint: You might need to use Exercise 1.

Exercise 3

1. Consider $\boldsymbol{A} \in S^n_{++}$ and matrix $\boldsymbol{\Delta} \in S^n_{+}$. Prove that $(\boldsymbol{A} + \boldsymbol{\Delta})^{-1} \preceq \boldsymbol{A}^{-1}$.
2. Let T be a convex set. We say that a function $f : T \to \mathbb{R}^{n \times n}$, is operator convex if for any two matrices $\boldsymbol{A}, \boldsymbol{B} \in T$ and any $\theta \in [0, 1]$

$$f\left(\theta\boldsymbol{X} + (1-\theta)\ \boldsymbol{Y}\right) \preceq \theta f\left(\boldsymbol{X}\right) + (1-\theta)\, f\left(\boldsymbol{Y}\right).$$

   Prove that $f(\boldsymbol{X}) = \boldsymbol{X}^{-1}$ is operator convex over the set $T = S^n_{++}$.

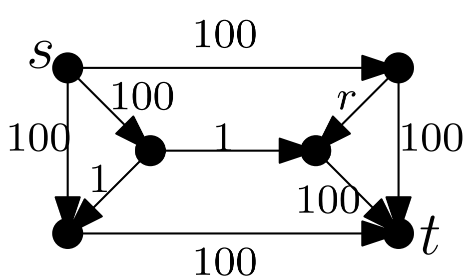
   Hint: You could first show that operator convexity is implied by the second directional derivative $D^2 f(\boldsymbol{X})[\boldsymbol{Y}, \boldsymbol{Y}]$ being positive semi-definite for all $\boldsymbol{Y} \in S^n$ and $\boldsymbol{X} \in S^n_{++}$.

Exercise 4.A: Convergence of Ford-Fulkerson

Show that the Ford-Fulkerson algorithm may not terminate; moreover, it may converge toward a value not equal to the value of the maximum flow.

Hint: You might use the graph below with the given capacities, where $r = \frac{\sqrt{5}-1}{2}$ (which implies that $r^2 = 1 - r$).

Warning: This exercise is quite unpleasant – you may want to focus on the other ones, and only do this one if you have time to spare.



Exercise 4.B: Iterative Refinement for Maximum Flow

Suppose we have an algorithm FlowRefine, which given a maximum flow instance $G = (V, E, \boldsymbol{c})$ with source $s \in V$ and sink $t \in V$ returns a feasible s-t flow $\tilde{\boldsymbol{f}}$, i.e. $\boldsymbol{B}\tilde{\boldsymbol{f}} = F(-\boldsymbol{\chi}_s + \boldsymbol{\chi}_t)$ for some F, and $\boldsymbol{0} \leq \tilde{\boldsymbol{f}} \leq \boldsymbol{c}$, and $\tilde{\boldsymbol{f}}$ is guaranteed to route at least half the maximum flow, i.e. $F = \mathrm{val}(\tilde{\boldsymbol{f}}) \geq 0.5\,\mathrm{val}(\boldsymbol{f}^*)$.

Suppose that the running time of FlowRefine is $O(|E|^c)$ for some constant $c \geq 1$.

Explain how we can use FlowRefine to find a flow $\hat{\boldsymbol{f}}$ that routes at least $(1-\epsilon)\,\mathrm{val}(\boldsymbol{f}^*)$ in time $O(|E|^c \log(1/\epsilon))$.

Exercise 5

Maximum flow is often introduced as a s-t problem in the following manner: Given a directed and capacitated graph $G = (V, E, \boldsymbol{c})$ with n vertices and m edges (assume integer capacities between 1 and n^{10}) and edge-vertex incidence matrix $\boldsymbol{B}$, a source $s \in V$ and a sink $t \in V$, computes the flow $\boldsymbol{f}$ that respects the capacities $\boldsymbol{0} \leq \boldsymbol{f} \leq \boldsymbol{c}$, satisfies flow conservation $(\boldsymbol{B}\boldsymbol{f})(v) =$ for $v \in V \setminus \{s, t\}$ and maximizes F where $(\boldsymbol{B}\boldsymbol{f})(s) = -F$ and $(\boldsymbol{B}\boldsymbol{f})(t) = F$.

1. Show that such an algorithm can be used to solve the maximum flow problem as introduced in the lecture, i.e. given a integral demand vector $\boldsymbol{d} \perp \boldsymbol{1}$ with entries between $-n^{10}$ and n^{10}, compute a flow that satisfies
$$\min_{\boldsymbol{0} \leq \boldsymbol{f}, \boldsymbol{B}\boldsymbol{f} = \boldsymbol{d}} \left\| \mathrm{diag}(\boldsymbol{c})^{-1} \boldsymbol{f} \right\|_\infty . \tag{1}$$
2. Show that the converse is also true, i.e. that a solution to (1) can be used to solve the s-t maximum flow problem.

Hint: There is always an integral maximum flow $\boldsymbol{f}$ in this setting.