Course Introduction

R. Kyng & M. Probst *Problem Set 1 — Tuesday, February 23rd*

The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all.

Exercise 1.

Let us define the *α-sub-level set* of a function $f : \mathbb{R}^n \to \mathbb{R}$ to be the set $S_\alpha \stackrel{\text{def}}{=} \{\boldsymbol{x} : f(\boldsymbol{x}) \leq \alpha\}$.

(i) Prove that if a function f is convex, then all its sub-level sets are convex sets.

(ii) Is it true that a function whose sub-levels sets are all convex is necessarily convex?

Exercise 2.

Recall the definition of Laplacian $\boldsymbol{L} = \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top$.

(i) We can also define Laplacian as $\boldsymbol{L} \stackrel{\text{def}}{=} \boldsymbol{D} - \boldsymbol{A}$, where $\boldsymbol{A}$ is the weighted adjacency matrix, i.e. $\boldsymbol{A}(u, v) = 1/\boldsymbol{r}(u, v)$, and $\boldsymbol{D} \stackrel{\text{def}}{=} \operatorname{diag}_{v \in V} \boldsymbol{w}(v)$ for $\boldsymbol{w}(v) := \sum_{(u,v) \in E} 1/\boldsymbol{r}(u, v)$. Prove that these two definitions are equivalent.

(ii) Given a function on the vertices, $\boldsymbol{x} \in \mathbb{R}^V$, the Laplacian quadratic form is

$$\boldsymbol{x}^\top \boldsymbol{L}\boldsymbol{x} = \sum_{(u,v) \in E} \frac{(\boldsymbol{x}(u) - \boldsymbol{x}(v))^2}{\boldsymbol{r}(u, v)}.$$

Prove the above equality and building on that, show that $\boldsymbol{L}$ is positive semi-definite.

(iii) What is the kernel of $\boldsymbol{L}$, which is denoted by $\operatorname{Ker}(\boldsymbol{L})$?

Exercise 3.

(i) Prove that for a matrix $\boldsymbol{A}$ we have $\operatorname{im}(\boldsymbol{A}) = \ker(\boldsymbol{A}^\top)^\perp$, where $\operatorname{im}(\boldsymbol{A})$ denotes the image of $\boldsymbol{A}$ and $\ker(\boldsymbol{A}^\top)^\perp$ is the orthogonal complement to $\ker(\boldsymbol{A}^\top)$.

(ii) Building on part (i), prove that in our flow problem, when the graph is connected, an electrical flow $\boldsymbol{f}$ routing $\boldsymbol{d}$ exists if and only if $\boldsymbol{1}^\top \boldsymbol{d} = 0$.

Exercise 4.

Define the gradient of a multivariate function $f : S \to \mathbb{R}$ for $S \subseteq \mathbb{R}^n$. Then, prove that the system of linear equations $\boldsymbol{L}\boldsymbol{x} = \boldsymbol{d}$ is the same as the system obtained by setting the gradient with respect to $\boldsymbol{x}$ of the function $c(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{L}\boldsymbol{x} - \boldsymbol{x}^\top \boldsymbol{d}$ equal to zero.

Exercise 5.

The goal of this exercise is to prove that

$$\max_{\boldsymbol{x}\in\mathbb{R}^V} \boldsymbol{x}^\top \boldsymbol{d} - \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{L}\boldsymbol{x} = \min_{\boldsymbol{f}\in\mathbb{R}^E} \frac{1}{2}\sum_e \boldsymbol{r}(e)\boldsymbol{f}(e)^2 \quad \text{s.t. } \boldsymbol{B}\boldsymbol{f} = \boldsymbol{d}.$$

We'll break that down into a few steps.

Let $\boldsymbol{f} \in \mathbb{R}^E$ be an arbitrary flow that satisfies $\boldsymbol{B}\boldsymbol{f} = \boldsymbol{d}$, i.e. it routes the demand $\boldsymbol{d}$. Let $\boldsymbol{x} \in \mathbb{R}^V$ be arbitrary voltages. *Arbitrary* means you cannot assume these are the electrical the electrical flow and voltages.

(i) Prove that

$$\frac{1}{2}\sum_e \boldsymbol{r}(e)\boldsymbol{f}(e)^2 = \boldsymbol{x}^\top \boldsymbol{d} - \left(\sum_{(u,v)\in E} (\boldsymbol{x}(u) - \boldsymbol{x}(v))(\boldsymbol{f}(u,v)) - \frac{1}{2}\boldsymbol{r}(u,v)\boldsymbol{f}(u,v)^2\right)$$

Hint: use that $\boldsymbol{x}^\top(\boldsymbol{B}\boldsymbol{f} - \boldsymbol{d}) = 0$.

(ii) Prove that

$$(\boldsymbol{x}(u) - \boldsymbol{x}(v))(\boldsymbol{f}(u,v)) - \frac{1}{2}\boldsymbol{r}(u,v)\boldsymbol{f}(u,v)^2 \leq \frac{1}{2}\frac{(\boldsymbol{x}(u) - \boldsymbol{x}(v))^2}{\boldsymbol{r}(u,v)}.$$

(iii) Conclude that $\frac{1}{2}\boldsymbol{f}^\top \boldsymbol{R}\boldsymbol{f} \geq \boldsymbol{x}^\top \boldsymbol{d} - \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{L}\boldsymbol{x}$.

(iv) Assume we are given $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{f}}$ such that

$$\boldsymbol{L}\tilde{\boldsymbol{x}} = \boldsymbol{d} \text{ and } \tilde{\boldsymbol{f}} = \boldsymbol{R}^{-1}\boldsymbol{B}^\top \tilde{\boldsymbol{x}}$$

Prove that $\boldsymbol{B}\tilde{\boldsymbol{f}} = \boldsymbol{d}$ and

$$\tilde{\boldsymbol{x}}^\top \boldsymbol{d} - \frac{1}{2}\tilde{\boldsymbol{x}}^\top \boldsymbol{L}\tilde{\boldsymbol{x}} = \frac{1}{2}\tilde{\boldsymbol{f}}^\top \boldsymbol{R}\tilde{\boldsymbol{f}}.$$

(v) Show

$$\tilde{\boldsymbol{x}} \in \arg\max_{\boldsymbol{x}\in\mathbb{R}^V} \boldsymbol{x}^\top \boldsymbol{d} - \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{L}\boldsymbol{x}$$

and

$$\tilde{\boldsymbol{f}} \in \arg\min_{\boldsymbol{f}\in\mathbb{R}^E} \frac{1}{2}\sum_e \boldsymbol{r}(e)\boldsymbol{f}(e)^2 \quad \text{s.t. } \boldsymbol{B}\boldsymbol{f} = \boldsymbol{d}.$$

Exercise 6.

Recall that the following theorem gives us a sufficient (though not necessary) condition for optimality.

Theorem (Extreme Value Theorem)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $\mathcal{F} \subseteq \mathbb{R}^n$ be nonempty, bounded, and closed. Then, the optimization problem $\min f(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{F}$ has an optimal solution.

Prove the above theorem. You might use the following two theorems.

Theorem (Bolzano-Weierstrass)**.** Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Theorem (Boundedness Theorem)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and $\mathcal{F} \subseteq \mathbb{R}^n$ be nonempty, bounded, and closed. Then f is bounded on $\mathcal{F}$.

Exercise 7.

Prove Taylor's Theorem.

Theorem (Taylor's Theorem, multivariate first-order remainder form)**.** If $f : S \to \mathbb{R}$ is continuously differentiable over $[\boldsymbol{x}, \boldsymbol{y}]$, then for some $\boldsymbol{z} \in [\boldsymbol{x}, \boldsymbol{y}]$,

$$f(\boldsymbol{y}) = f(\boldsymbol{x}) + \boldsymbol{\nabla} f(\boldsymbol{z})^\top (\boldsymbol{y} - \boldsymbol{x}).$$

Exercise 8.

Let $f_1(\boldsymbol{x}), \cdots, f_k(\boldsymbol{x})$ be a set of convex functions with the same domain and define

$$f(\boldsymbol{x}) \stackrel{\text{def}}{=} \max_{1 \le i \le k} f_i(\boldsymbol{x}).$$

Prove that $f(\boldsymbol{x})$ is convex.

Exercise 9.

Assume that $f(x, y)$ is a convex function and S is a convex non-empty set. Prove that

$$g(x) = \inf_{y \in S} f(x, y)$$

is convex, provided $g(x) > -\infty$ for all x.

Exercise 10.

For each function below, determine whether it is convex or not.

1. $f(x) = |x|^6$ on $x \in \mathbb{R}$
2. $f(x) = \exp(x)$ on $x \in (0, \infty)$
3. $f(x, y) = \sqrt{x + y}$ on $(x, y) \in (0, 1) \times (0, 1)$
4. $f(x, y) = xy$ on $(x, y) \in (-1, 1) \times (-1, 1)$