Advanced Graph Algorithms and Optimization	Spring 2021

Accelerated Gradient Descent, Spectral Graph Theory

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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all.

Exercise 1.

This exercise asks you to prove Claim 3.4.2 from Chapter 3, Section 3.4 on Accelerated Gradient Decscent.

Prove that

1. $\boldsymbol{v}_0 = \boldsymbol{x}_0 - \frac{a_0}{\sigma} \boldsymbol{\nabla} f(\boldsymbol{x}_0)$
2. $L_0 = f(\boldsymbol{x}_0) - \frac{a_0}{2\sigma} \left\| \boldsymbol{\nabla} f(\boldsymbol{x}_0) \right\|_2^2 - \frac{\sigma}{2a_0} \left\| \boldsymbol{x}^* - \boldsymbol{x}_0 \right\|_2^2$.

Exercise 2.

This exercise asks you to prove Claim 3.4.3 from Chapter 3, Section 3.4 on Accelerated Gradient Decscent.

Prove that

1. $m_i(\boldsymbol{v}) = m_i(\boldsymbol{v}_i) + \frac{\sigma}{2} \left\| \boldsymbol{v} - \boldsymbol{v}_i \right\|_2^2$
2. $m_{i+1}(\boldsymbol{v}) = m_i(\boldsymbol{v}) + a_{i+1} f(\boldsymbol{x}_{i+1}) + \langle a_{i+1} \boldsymbol{\nabla} f(\boldsymbol{x}_{i+1}), \boldsymbol{v} - \boldsymbol{x}_{i+1} \rangle$
3. $\boldsymbol{v}_{i+1} = \boldsymbol{v}_i - \frac{a_{i+1}}{\sigma} \boldsymbol{\nabla} f(\boldsymbol{x}_{i+1})$.

Exercise 3.

This exercise is a straggler from last week, where we studied convexity. It will teach you about Jensen's inequality, one of the most important inequalities that we use when studying convex functions.

1. Assume that $S \subseteq \mathbb{R}^n$ is a convex set and that the function $f : S \to \mathbb{R}$ is convex. Suppose that $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n \in S$ and $\theta_1, \cdots, \theta_n \geq 0$ with $\theta_1 + \cdots + \theta_n = 1$. Prove that

   $$f(\theta_1 \boldsymbol{x}_1 + \cdots + \theta_n \boldsymbol{x}_n) \leq \theta_1 f(\boldsymbol{x}_1) + \cdots + \theta_n f(\boldsymbol{x}_n).$$

   Remark. This is typically known as Jensen's inequality and can be extended to infinite sums. If $\mathcal{D}$ is a probability distribution on S, and $\boldsymbol{X} \sim \mathcal{D}$, then

   $$f(\mathbb{E}[\boldsymbol{X}]) \leq \mathbb{E}[f(\boldsymbol{X})]$$

whenever both integrals are finite.

2. Prove that
$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i.$$

3. Prove that
$$\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}} \leq \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.$$

Exercise 4

Let P_n be the path from vertex 1 to n and $G_{1,n}$ be the graph with only the edge between vertex 1 and n. Furthermore, assume that the edge between vertex i and $i+1$ has positive weight w_i for $1 \leq i \leq n-1$. Prove that
$$G_{1,n} \preceq \left(\sum_{i=1}^{n-1}\frac{1}{w_i}\right)\sum_{i=1}^{n-1} w_i G_{i,i+1}.$$

Exercise 5

In Chapter 4, we proved that
$$\lambda_2(T_d) \geq \frac{1}{(n-1)\log_2 n}.$$

Improve this bound to $\lambda_2(T_d) \geq 1/cn$ for some constant $c > 0$.

Hint: Use the result of previous exercise.