Advanced Graph Algorithms and Optimization Spring 2021

Spectral Graph Theory

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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all.

Exercise 1.

Find the conductance $\phi \in (0, 1]$ for the following graphs:

1. the complete graph K_n over n vertices.
2. the path graph P_n over n vertices.

Exercise 2.

Bound roughly how long it takes for a lazy random walk with initial distribution $\boldsymbol{p}_0$ to have $\|\boldsymbol{p}_t - \pi\|_\infty \leq \epsilon$ (for some parameter $\epsilon > 0$) for

1. the complete graph K_n over n vertices.
2. the path graph P_n over n vertices.

Hint: Use conductances from the last exercise and apply Cheeger's inequality.

Exercise 3.

Show that $\lambda_2(\boldsymbol{L}) \neq 0$ if and only if G is connected. Argue that the same applies for $\boldsymbol{N}$.

Exercise 4.

A quite related concept to conductance is *sparsity*: we define the sparsity of a cut $\emptyset \subset S \subset V$ by

$$\sigma(S) = \frac{|E(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}}.$$

An alternative version of Cheeger's inequality relates the second eigenvalue of $\boldsymbol{L}$ (not $\boldsymbol{N}$) to the sparsity of the graph $\sigma(G) = \min_{\emptyset \subset S \subset V} \sigma(S)$:

$$\frac{\lambda_2(\boldsymbol{L})}{2} \leq \sigma(G) \leq \sqrt{2 d_{max} \cdot \lambda_2(\boldsymbol{L})}$$

where d_{max} is the maximum degree of any vertex in the graph.

Prove the lower bound on $\sigma(G)$, i.e. that $\frac{\lambda_2(\boldsymbol{L})}{2} \leq \sigma(G)$.

Hint: Follow closely the proof of of the lower bound in Cheeger's inequality and try to understand what has to be adapted.

Exercise 5.

In the lecture, we skipped various steps in the proof of Cheeger's inequality. Show that

1. $\boldsymbol{N}$ is symmetric and in fact PSD. We recommend to prove this by proving the following stronger statement: for any matrix $\boldsymbol{A}$ that is PSD, and any matrix $\boldsymbol{C}$, we have that $\boldsymbol{C}^\top \boldsymbol{A} \boldsymbol{C}$ is PSD.

2. Show that the normalization of $\boldsymbol{z}$ in the upper bound proof of Cheeger's inequality can only make the ratio we are interested in smaller. I.e. prove that $\frac{\boldsymbol{z}^\top \boldsymbol{L} \boldsymbol{z}}{\boldsymbol{z}^\top \boldsymbol{D} \boldsymbol{z}} \geq \frac{\boldsymbol{z}_{sc}^\top \boldsymbol{L} \boldsymbol{z}_{sc}}{\boldsymbol{z}_{sc}^\top \boldsymbol{D} \boldsymbol{z}_{sc}}$.

   Hint: Argue first about the transformation of $\boldsymbol{z}$ to $\boldsymbol{z}_c$. One way of relating their denominator is by minimizing over all choices of $\boldsymbol{z}_c$ for c. For $\boldsymbol{z}_c$ and $\boldsymbol{z}_{sc}$ you should be able to prove an equality.

3. We have also skipped proving that τ is sampled according to a valid probability distribution: Show that $\int_\tau \mathbb{P}[\tau = \ell] \, d\tau = 1$.

   Hint: Recall the properties of $\boldsymbol{z}_{sc}$.

4. Show that

$$\mathbb{E}_\tau[|E(S_\tau, V \setminus S_\tau)|] \leq \sum_{\{i,j\} \in E} |\boldsymbol{z}_{sc}(i) - \boldsymbol{z}_{sc}(j)| \cdot (|\boldsymbol{z}_{sc}(i)| + |\boldsymbol{z}_{sc}(j)|)$$

   by concluding the argument in the proof.

5. Standard Probabilistic Method: Consider a random variable X with a discrete distribution and let Ω be the sample space. Argue that there exists an $\omega \in \Omega$ with $X(\omega) \geq \mathbb{E}[X]$.

   Hint: Recall the definition of expectation of a discrete random variable.

6. Using the probabilisitic method for Cheeger's Inequality: recall that in our proof, we want to argue that $\frac{\mathbb{E}_\tau[\boldsymbol{1}_S^\top \boldsymbol{L} \boldsymbol{1}_S]}{\mathbb{E}_\tau[\boldsymbol{1}_S^\top \boldsymbol{D} \boldsymbol{1}_S]} \leq \sqrt{2 \cdot \frac{\boldsymbol{z}_{sc}^\top \boldsymbol{L} \boldsymbol{z}_{sc}}{\boldsymbol{z}_{sc}^\top \boldsymbol{D} \boldsymbol{z}_{sc}}}$ implies that there exists an S with $\frac{\boldsymbol{1}_S^\top \boldsymbol{L} \boldsymbol{1}_S}{\boldsymbol{1}_S^\top \boldsymbol{D} \boldsymbol{1}_S} \leq \sqrt{2 \cdot \frac{\boldsymbol{z}_{sc}^\top \boldsymbol{L} \boldsymbol{z}_{sc}}{\boldsymbol{z}_{sc}^\top \boldsymbol{D} \boldsymbol{z}_{sc}}}$. There are two ways to prove this (feel free to choose just one):

   (a) you can prove this claim by considering $\mathbb{E}_\tau \left[\boldsymbol{1}_S^\top \boldsymbol{L} \boldsymbol{1}_S\right] \leq \sqrt{2 \cdot \frac{\boldsymbol{z}_{sc}^\top \boldsymbol{L} \boldsymbol{z}_{sc}}{\boldsymbol{z}_{sc}^\top \boldsymbol{D} \boldsymbol{z}_{sc}}} \cdot \mathbb{E}_\tau \left[\boldsymbol{1}_S^\top \boldsymbol{D} \boldsymbol{1}_S\right]$. Use only linearity of expectation to obtain an expression with a single $\mathbb{E}_\tau$ and apply the probabilistic method, or

   (b) you can prove that for any two discrete random variables $X, Y > 0$ with the same distribution, we have that there exists an $\omega \in \Omega$ with

$$\frac{X(\omega)}{Y(\omega)} \leq \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}.$$

Exercise 6.

The bound obtained in Cheeger's inequality is indeed tight. Prove that:

1. Let G be the graph consisting of two vertices connected by a single edge of unit weight. Prove that $\phi(G) = \lambda_2(\boldsymbol{N})/2$ and therefore that the lower bound of Cheeger's inequality is tight.
2. To show that the line graph proves that the upper bound of Cheeger's Inequality is asymptotically tight (i.e. up to constant factors).

Exercise 7.

Sparse Expanders: In random graph theory, the graph over n vertices where each edge between two endpoints is present independently with probability p is denoted $G(n, p)$.

Show that for $p = \Omega(\log n/n)$, that $G(n, p)$ is a $\Omega(1)$-expander with high probability (it is up to you to fix large constants). Take the following steps:

1. Prove that with high probability, $\boldsymbol{d}(u) = \Theta(pn)$ for all vertices $u \in V(G(n, p))$.
2. For each set S of $k \leq n/2$ vertices, argue that
$$\mathbb{P}[|E(S, V \setminus S)| = \Theta(kpn)] > 1 - n^{-c \cdot k}$$
for any large constant $c > 0$.
3. Observe that there are at most $\binom{n}{k}$ sets of vertices S of size k. Conclude that $G(n, p)$ is with high probability a $\Omega(1)$-expander.