Advanced Graph Algorithms and Optimization	Spring 2021

Spectral Graph Theory

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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all.

Througout these exercises, we will use the following notation:

- S^n is the set of symmetric real matrices $n \times n$ matrices.
- S^n_+ is the set of positive semi-definite $n \times n$ matrices.
- S^n_{++} is the set of positive definite $n \times n$ matrices.

Whenever we say a matrix is positive semi-definite or positive definite, we require it to be real and symmetric.

Exercise 1.

1. Show that there exist two matrices $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$ such that $\boldsymbol{A} \preceq \boldsymbol{B}$ but $\boldsymbol{A}^2 \not\preceq \boldsymbol{B}^2$.
2. Let $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$, and assume $\boldsymbol{A} \preceq \boldsymbol{B}$. Prove that $\boldsymbol{B}^{-1} \preceq \boldsymbol{A}^{-1}$.
   Hint: It might help to first prove that for a matrix $\boldsymbol{C} \in \mathbb{R}^{n \times n}$, we have $\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^\top \preceq \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^\top$.

Exercise 2.

For a matrix $\boldsymbol{Z}$ to be the pseudoinverse of a symmetric matrix $\boldsymbol{M}$, you need to show that

1. $\boldsymbol{Z}^\top = \boldsymbol{Z}$.
2. $\boldsymbol{Z}\boldsymbol{v} = \boldsymbol{0}$ for $\boldsymbol{v} \in \ker(\boldsymbol{M})$.
3. $\boldsymbol{M}\boldsymbol{Z}\boldsymbol{v} = \boldsymbol{v}$ for $\boldsymbol{v} \in \ker(\boldsymbol{M})^\perp$.

Prove that if $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are both the pseudo-inverse of symmetric matrix $\boldsymbol{M}$, then $\boldsymbol{Z} = \boldsymbol{Y}$, i.e. the pseudo-inverse is unique.

Exercise 3.

Let $\boldsymbol{M} = \boldsymbol{X}\boldsymbol{Y}\boldsymbol{X}^\top$ for some $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n \times n}$, where $\boldsymbol{X}$ is invertible and $\boldsymbol{M}$ is symmetric. Furthermore, consider the spectral decomposition of $\boldsymbol{M} = \sum_{i=1}^n \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^\top$. Then, we define $\boldsymbol{\Pi}_{\boldsymbol{M}} = \sum_{i, \lambda_i \neq 0} \boldsymbol{v}_i \boldsymbol{v}_i^\top$. $\boldsymbol{\Pi}_{\boldsymbol{M}}$ is the orthogonal projection onto the image of $\boldsymbol{M}$: It has the property that for $\boldsymbol{v} \in \mathrm{im}(\boldsymbol{M})$, $\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{v}$ and for $\boldsymbol{v} \in \ker(\boldsymbol{M})$, $\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{0}$.

Prove that

$$\boldsymbol{Z} = \boldsymbol{\Pi}_M (\boldsymbol{X}^\top)^{-1} \boldsymbol{Y}^+ \boldsymbol{X}^{-1} \boldsymbol{\Pi}_M$$

is the pseudoinverse of $\boldsymbol{M}$.

Exercise 4.

Suppose that a weighted graph G is a ϕ-expander, with Laplacian $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$.

1. Prove that for any $\boldsymbol{z} \perp \mathbf{1}$,

   $$\boldsymbol{z}^\top \boldsymbol{L}^\dagger \boldsymbol{z} \preceq 2\phi^{-2} \boldsymbol{z}^\top \boldsymbol{D}^{-1} \boldsymbol{z}.$$

   Hint: Use the previous exercise. Be careful about handling the null space.

2. Use the statement above to give an upper bound on the effective resistance between any two vertices u, v of G.

Exercise 5.

Let the graphs below be unweighted and undirected.

1. Calculate the expected hitting time $\mathbb{E}[H_{1,n}]$ of the random walk on the path graph P_n starting in vertex 1 until it reaches vertex n.

2. Calculate the expected hitting time $\mathbb{E}[H_{a,b}]$ for any two vertices $a \neq b \in V$ in the complete graphs K_{n+1}.

3. The graph consisting of K_n and the path P_n joined at an arbitrary vertex of K_n and the first vertex on P_n is often called the Lollipop graph $L_{n,n}$. Show that there exists a set of vertices $a, b \in V(L_{n,n})$, with $\mathbb{E}[H_{a,b}] \neq \mathbb{E}[H_{b,a}]$.