Advanced Graph Algorithms and Optimization Spring 2021

(Mostly) Convex Duality

R. Kyng & M. Probst *Problem Set 9 — Tuesday, May 11th*

The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all.

Exercise 1: Different Duals

Let $G = (V, E)$ be a directed graph with capacities $\boldsymbol{c} \in \mathbb{R}^E \geq \boldsymbol{0}$, and edge-vertex incidence matrix $\boldsymbol{B}$, and consider a demand vector $\boldsymbol{d} \in \mathbb{R}^V$ with $\boldsymbol{d} \perp \boldsymbol{1}$.

Let us try to minimize $\left\|\boldsymbol{C}^{-1}\boldsymbol{f}\right\|_\infty$, where $\boldsymbol{C} = \operatorname{diag}_{e\in E} \boldsymbol{c}(e)$. This leads to an optimization problem

$$\begin{aligned} \min_{\boldsymbol{f}\in\mathbb{R}^E_{\geq 0}} & \left\|\boldsymbol{C}^{-1}\boldsymbol{f}\right\|_\infty \\ \text{s.t. } & \boldsymbol{B}\boldsymbol{f} = \boldsymbol{d} \end{aligned}$$

- Compute the dual of this problem. Note that we encoded the "constraint" $\boldsymbol{f} \geq \boldsymbol{0}$ in the domain of the variable $\boldsymbol{f}$. This means that there will *not* be a dual variable associated with this constraint.
- Does Strong Duality hold for this pair of primal and dual problems?

Consider also a variant of this problem, where we instead treat $\boldsymbol{f} \geq \boldsymbol{0}$ as an explicit constraint:

$$\begin{aligned} \min_{\boldsymbol{f}\in\mathbb{R}^E} & \left\|\boldsymbol{C}^{-1}\boldsymbol{f}\right\|_\infty \\ \text{s.t. } & \boldsymbol{B}\boldsymbol{f} = \boldsymbol{d} \\ & \boldsymbol{f} \geq \boldsymbol{0} \end{aligned}$$

This problem has a different dual problem because we made $\boldsymbol{f} \geq \boldsymbol{0}$ an explicit constraint.

- Compute the dual of this variant of the problem.
- Explain how, given an optimal solution to the first dual problem, we can compute an optimal solution to the second dual problem.

Exercise 2: A "Broken" Dual

Consider the following optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}, y\in\mathbb{R}_{>0}} & \ e^{-x} \\ \text{s.t. } & x^2/y \leq 0. \end{aligned}$$

- Compute the dual program.
- What is the optmal value of the primal program? And of the dual program? Does strong duality hold? Does Slater's condition hold?

Exercise 3: Norms and a Lagragian

Suppose $1 < q < p < \infty$. In this exercise, we want to prove that for $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$\|\boldsymbol{x}\|_q \leq n^{1/q-1/p} \|\boldsymbol{x}\|_p . \tag{1}$$

Consider the following optimization problem:

$$\max_{\boldsymbol{x}\in\mathbb{R}^n : \|\boldsymbol{x}\|_p^p \leq 1} \|\boldsymbol{x}\|_q^q$$

- Is this a convex optimization problem?
- Informally, explain why, at any maximizing $\boldsymbol{x}$ for the above problem, there should exist a $\lambda > 0$ such that
$$\boldsymbol{\nabla}_{\boldsymbol{x}} \left(\|\boldsymbol{x}\|_q^q - \lambda \|\boldsymbol{x}\|_p^p \right) = \boldsymbol{0}.$$
- Use the existence of such a λ to prove Equation (**??**).

Exercise 4: Flows and Voltages and Other Powers

Consider a connected undirected graph $G = (V, E)$ with resistances $\boldsymbol{r} \in \mathbb{R}^E$ and edge-vertex incidence matrix $\boldsymbol{B}$, and a demand vector $\boldsymbol{d} \in \mathbb{R}^V$ with $\boldsymbol{d} \perp \boldsymbol{1}$.

Given some $p > 1$, we'll look at the flow optimization problem

$$\min_{\boldsymbol{f}\in\mathbb{R}^E} \sum_e \boldsymbol{r}(e) \frac{1}{p} |\boldsymbol{f}(e)|^p$$
$$\text{s.t. } \boldsymbol{B}\boldsymbol{f} = \boldsymbol{d}.$$

- Is the above optimization problem convex?
- Does Slater's condition hold for the problem?
- What is the dual problem for the problem? Define $q > 0$ to be the number such that $1 = \frac{1}{q} + \frac{1}{p}$. Try to find a clean expression of the dual. Writing the expression in terms of q instead of p will simplify it.

Suppose our instance of the optimization problem as an optimal flow solution $\boldsymbol{f}^*$. Let $\alpha = \sum_e \boldsymbol{r}(e) \frac{1}{p} |\boldsymbol{f}^*(e)|^p$ be the optimal problem value. Suppose that for some particular edge $\hat{e}$ we have

$$\gamma = \frac{\boldsymbol{r}(\hat{e}) \frac{1}{p} |\boldsymbol{f}^*(\hat{e})|^p}{\alpha}.$$

Now, consider a modified instance with resistances $\tilde{\boldsymbol{r}} \in \mathbb{R}^E$ given by

$$\tilde{\boldsymbol{r}}(e) = \begin{cases} \boldsymbol{r}(e) & \text{for } e \neq \hat{e} \\ 2^{p-1}\boldsymbol{r}(e) & \text{for } e = \hat{e} \end{cases}$$

That is, we increase the resistance on edge $\hat{e}$ by a factor 2^{p-1}. Let $\tilde{\alpha}$ denote optimal value of program with the new resistances $\tilde{\boldsymbol{r}}$. Prove that

$$\tilde{\alpha} \geq \left(1 + \frac{p-1}{2}\gamma\right)\alpha.$$

Hint: use the dual problem!