

## Spectral Graph Theory

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Problem Set 4 — Wednesday, March 15

These exercises will not count toward your grade, but you are encouraged to solve them all. This exercise sheet contains exercises relating to lectures in Week 4.

To get feedback, you must hand in your solutions by 23:59 on March 23. Both hand-written and L<sup>A</sup>T<sub>E</sub>X solutions are acceptable, but we will only attempt to read legible text.

**Exercise 1**

Bound roughly how long it takes for a lazy random walk with initial distribution  $\mathbf{p}_0 = \mathbf{1}_a$  for some  $a \in V$  to have  $\|\mathbf{p}_t - \pi\|_\infty \leq \epsilon$  (for some parameter  $\epsilon > 0$ ) for

1. the complete graph  $K_n$  over  $n$  vertices.
2. the path graph  $P_n$  over  $n$  vertices.

*Hint: Use conductances from the last exercise and apply Cheeger's inequality.*

**Exercise 2**

Let the graphs below be unweighted and undirected.

1. Calculate the expected hitting time  $\mathbb{E}[H_{1,n}]$  of the random walk on the path graph  $P_n$  starting in vertex 1 until it reaches vertex  $n$ .
2. Calculate the expected hitting time  $\mathbb{E}[H_{a,b}]$  for any two vertices  $a \neq b \in V$  in the complete graphs  $K_{n+1}$ .
3. The graph consisting of  $K_n$  and the path  $P_n$  joined at an arbitrary vertex of  $K_n$  and the first vertex on  $P_n$  is often called the Lollipop graph  $L_{n,n}$ . Show that there exists a set of vertices  $a, b \in V(L_{n,n})$ , with  $\mathbb{E}[H_{a,b}] \neq \mathbb{E}[H_{b,a}]$ .

**Exercise 3**

The bound obtained in Cheeger's inequality is indeed tight. Prove that:

1. Let  $G$  be the graph consisting of two vertices connected by a single edge of unit weight. Prove that  $\phi(G) = \lambda_2(\mathbf{N})/2$  and therefore that the lower bound of Cheeger's inequality is tight.
2. To show that the line graph proves that the upper bound of Cheeger's Inequality is asymptotically tight (i.e. up to constant factors).

## Exercise 4

Sparse Expanders: In random graph theory, the graph over  $n$  vertices where each edge between two endpoints is present independently with probability  $p$  is denoted  $G(n, p)$ .

Show that for  $p = \Omega(\log n/n)$ , that  $G(n, p)$  is a  $\Omega(1)$ -expander with high probability (it is up to you to fix large constants). Take the following steps:

1. Prove that with high probability,  $\mathbf{d}(u) = \Theta(pn)$  for all vertices  $u \in V(G(n, p))$ .
2. For each set  $S$  of  $k \leq n/2$  vertices, argue that

$$\mathbb{P}[|E(S, V \setminus S)| = \Theta(kpn)] > 1 - n^{-c \cdot k}$$

for any large constant  $c > 0$ .

3. Observe that there are at most  $\binom{n}{k}$  sets of vertices  $S$  of size  $k$ . Conclude that  $G(n, p)$  is with high probability a  $\Omega(1)$ -expander.

## Exercise 5

Let  $G = (V, E)$  be a connected, undirected graph. In this problem, you will show that there is an algorithm that computes a  $\phi$ -expander decomposition  $X_1, X_2, \dots, X_k$  for  $G$  of quality  $q = O(\phi^{-1/2} \cdot \log n)$  in time  $O(m \log^c n)$  for some constant  $c$ . We let  $\mathbf{N}$  denote the normalized Laplacian in this exercise, defined by  $\mathbf{N} = \mathbf{D}^{-1/2} \mathbf{L} \mathbf{D}^{-1/2}$ . Assume that you are given an algorithm  $\text{CERTIFYORCUT}(G, \phi)$  that given a graph  $G$  and a parameter  $\phi$  either:

- *Certifies* that  $G$  is a  $\phi$ -expander, or
- Presents a *cut*  $S$  such that  $\phi(S) = O(\sqrt{\phi})$ .

The algorithm  $\text{CERTIFYORCUT}(G, \phi)$  runs in time  $O(m \log^{c'} n)$  for  $c' > 0$ .

1. Show that there is an algorithm that uses  $\text{CERTIFYORCUT}(G, \phi)$  and computes a  $\phi$ -expander decomposition of quality  $O(\phi^{-1/2} \cdot \log n)$  in time  $O(mn \cdot \log^{c'} n)$ .
2. Show that in  $O(mn \cdot \log^{c'} n)$  time, you can implement a procedure  $\text{CERTIFYORLARGECUT}(G, \phi)$  that outputs a set  $S$  (possibly empty) with  $\phi(S) = O(\sqrt{\phi})$  such that either
  - $G[V \setminus S]$  is a  $\phi$ -expander and  $\text{vol}_G(V \setminus S) \geq \frac{1}{3}m$ , or
  - $\min\{\text{vol}_G(S), \text{vol}_G(V \setminus S)\} \geq \frac{1}{3}m$ .

*Comment: You may use the following claim.*

**Claim:** Given a set  $S \subseteq V$  of conductance  $\phi_G(S) \leq \phi$  and a set  $S' \subseteq V \setminus S$  in  $G[V \setminus S]$  with conductance  $\phi_{G[V \setminus S]}(S') \leq \phi$  such that  $\text{vol}_G(S \cup S') \leq \text{vol}_G(V)/2$ , then  $\phi_G(S \cup S') \leq \phi$ .

3. **(BONUS)** Using a local version of random walks and heavy randomization, it turns out that  $\text{CERTIFYORLARGECUT}(G, \phi)$  can be implemented in  $O(m \log^{c''} n)$  time<sup>1</sup>. Show that this implies an  $O(m \log^{c''+1} n)$  time algorithm to compute a  $\phi$ -expander decomposition.

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<sup>1</sup>Note that this is slightly idealized for ease of presentation.