Advanced Graph Algorithms and Optimization	Spring 2023
Spectral Graph Theory	
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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all. This exercise sheet has exercises related to week 5. We encourage you to start early so you have time to go through everything.

To get feedback, you must hand in your solutions by 23:59 on March 30. Both hand-written and LaTeX solutions are acceptable, but we will only attempt to read legible text.

Notation

Througout the following exercises, we will use the following notation:

- S^n is the set of symmetric real matrices $n \times n$ matrices.
- S_+^n is the set of positive semi-definite $n \times n$ matrices.
- S_{++}^n is the set of positive definite $n \times n$ matrices.

Whenever we say a matrix is positive semi-definite or positive definite, we require it to be real and symmetric.

Exercise 1

1. Show that there exist two matrices $\boldsymbol{A}, \boldsymbol{B} \in S_{++}^n$ such that $\boldsymbol{A} \preceq \boldsymbol{B}$ but $\boldsymbol{A}^2 \npreceq \boldsymbol{B}^2$.
2. Let $\boldsymbol{A}, \boldsymbol{B} \in S_{++}^n$, and assume $\boldsymbol{A} \preceq \boldsymbol{B}$. Prove that $\boldsymbol{B}^{-1} \preceq \boldsymbol{A}^{-1}$.
   Hint: It might help to first prove that for a matrix $\boldsymbol{C} \in \mathbb{R}^{n\times n}$, we have $\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^\top \preceq \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^\top$.

Exercise 2

For a matrix $\boldsymbol{Z}$ to be the pseudoinverse of a symmetric matrix $\boldsymbol{M}$, you need to show that

1. $\boldsymbol{Z}^\top = \boldsymbol{Z}$.
2. $\boldsymbol{Z}\boldsymbol{v} = \boldsymbol{0}$ for $\boldsymbol{v} \in \ker(\boldsymbol{M})$.
3. $\boldsymbol{M}\boldsymbol{Z}\boldsymbol{v} = \boldsymbol{v}$ for $\boldsymbol{v} \in \ker(\boldsymbol{M})^\perp$.

Prove that if $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are both the pseudo-inverse of symmetric matrix $\boldsymbol{M}$, then $\boldsymbol{Z} = \boldsymbol{Y}$, i.e. the pseudo-inverse is unique.

Exercise 3

Let $\boldsymbol{M} = \boldsymbol{X}\boldsymbol{Y}\boldsymbol{X}^\top$ for some $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n\times n}$, where $\boldsymbol{X}$ is invertible and $\boldsymbol{M}$ is symmetric. Furthermore, consider the spectral decomposition of $\boldsymbol{M} = \sum_{i=1}^n \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^\top$. Then, we define $\boldsymbol{\Pi}_{\boldsymbol{M}} = \sum_{i,\lambda_i \neq 0} \boldsymbol{v}_i \boldsymbol{v}_i^\top$. $\boldsymbol{\Pi}_{\boldsymbol{M}}$ is the orthogonal projection onto the image of $\boldsymbol{M}$: It has the property that for $\boldsymbol{v} \in \mathrm{im}(\boldsymbol{M})$, $\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{v}$ and for $\boldsymbol{v} \in \ker(\boldsymbol{M})$, $\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{0}$.

Prove that

$$\boldsymbol{Z} = \boldsymbol{\Pi}_{\boldsymbol{M}}(\boldsymbol{X}^\top)^{-1}\boldsymbol{Y}^+\boldsymbol{X}^{-1}\boldsymbol{\Pi}_{\boldsymbol{M}}$$

is the pseudoinverse of $\boldsymbol{M}$.

Exercise 4

Suppose that a weighted graph G is a ϕ-expander, with Laplacian $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$.

1. Prove that for any $\boldsymbol{z} \perp \boldsymbol{1}$,
$$\boldsymbol{z}^\top \boldsymbol{L}^\dagger \boldsymbol{z} \preceq 2\phi^{-2}\boldsymbol{z}^\top \boldsymbol{D}^{-1}\boldsymbol{z}.$$
Hint: Use the result from Exercise 3 in Problem Set 5.

2. Use the statement above to give an upper bound on the effective resistance between any two vertices u, v of G.