Advanced Graph Algorithms and Optimization	Spring 2023

Spectral Graph Theory

R. Kyng & M. Probst Gutenberg — *Problem Set 5— Wednesday, March 22nd*

The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all. This exercise sheet has exercises related to week 5. We encourage you to start early so you have time to go through everything.

To get feedback, you must hand in your solutions by 23:59 on March 30. Both hand-written and LaTeX solutions are acceptable, but we will only attempt to read legible text.

Notation

Througout the following exercises, we will use the following notation:

- S^n is the set of symmetric real matrices $n \times n$ matrices.
- S^n_+ is the set of positive semi-definite $n \times n$ matrices.
- S^n_{++} is the set of positive definite $n \times n$ matrices.

Whenever we say a matrix is positive semi-definite or positive definite, we require it to be real and symmetric.

Exercise 1

1. Show that there exist two matrices $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$ such that $\boldsymbol{A} \preceq \boldsymbol{B}$ but $\boldsymbol{A}^2 \not\preceq \boldsymbol{B}^2$.
2. Let $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$, and assume $\boldsymbol{A} \preceq \boldsymbol{B}$. Prove that $\boldsymbol{B}^{-1} \preceq \boldsymbol{A}^{-1}$.
   Hint: It might help to first prove that for a matrix $\boldsymbol{C} \in \mathbb{R}^{n \times n}$, we have $\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^\top \preceq \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^\top$.

Solution

1. Suppose that
   $$\boldsymbol{A} := \begin{bmatrix} 26 & 5 \\ 5 & 2 \end{bmatrix}$$
   and
   $$\boldsymbol{B} := \begin{bmatrix} 51 & 0 \\ 0 & 3 \end{bmatrix}.$$
   Consider an arbitrary vector $\boldsymbol{x} \in \mathbb{R}^2$. Then, we have that
   $$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = 26\boldsymbol{x}(1)^2 + 10\boldsymbol{x}(1)\boldsymbol{x}(2) + 2\boldsymbol{x}(2)^2.$$

and

$$\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} = 51\boldsymbol{x}(1)^2 + 3\boldsymbol{x}(2)^2.$$

Therefore, we get

$$\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} - \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = 25\boldsymbol{x}(1)^2 + \boldsymbol{x}(2)^2 - 10\boldsymbol{x}(1)\boldsymbol{x}(2) = (5\boldsymbol{x}(1) - \boldsymbol{x}(2))^2 \geq 0$$

which implies that $\boldsymbol{A} \preceq \boldsymbol{B}$.

Furthermore, $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$ because for any $\boldsymbol{x} \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$,

$$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} = 26\boldsymbol{x}(1)^2 + 10\boldsymbol{x}(1)\boldsymbol{x}(2) + 2\boldsymbol{x}(2)^2 = (5\boldsymbol{x}(1) + \boldsymbol{x}(2))^2 + \boldsymbol{x}(1)^2 + \boldsymbol{x}(2)^2 > 0.$$

On the other hand, for $\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we have $\boldsymbol{x}^\top \boldsymbol{A}^2\boldsymbol{x} = 29$ and $\boldsymbol{x}^\top \boldsymbol{B}^2\boldsymbol{x} = 9$ which implies that $\boldsymbol{A}^2 \not\preceq \boldsymbol{B}^2$.

2. Let $\boldsymbol{A}, \boldsymbol{B} \in S^n_{++}$, and assume that $\boldsymbol{A} \preceq \boldsymbol{B}$. Let us first prove that for any $\boldsymbol{C} \in \mathbb{R}^{n\times n}$,

$$\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^\top \preceq \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^\top. \tag{1}$$

For $\boldsymbol{x} \in \mathbb{R}^n$, we let $\boldsymbol{y} = \boldsymbol{C}^\top\boldsymbol{x}$. Then, we get

$$\boldsymbol{x}^\top \left(\boldsymbol{C}\boldsymbol{A}\boldsymbol{C}^\top - \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^\top\right)\boldsymbol{x} = \left(\boldsymbol{C}^\top\boldsymbol{x}\right)^\top (\boldsymbol{A} - \boldsymbol{B})\left(\boldsymbol{C}^\top\boldsymbol{x}\right) = \boldsymbol{y}^\top (\boldsymbol{A} - \boldsymbol{B})\,\boldsymbol{y} \leq 0.$$

Now, we prove that $\boldsymbol{B}^{-1} \preceq \boldsymbol{A}^{-1}$. We know that $\boldsymbol{B}^{-\frac{1}{2}}$ exists and $\boldsymbol{0} \prec \boldsymbol{B}^{-\frac{1}{2}}$ since $\boldsymbol{0} \prec \boldsymbol{B}$. By applying Equation (1), we have

$$\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{B}^{-\frac{1}{2}} \preceq \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{B}^{-\frac{1}{2}} = \boldsymbol{I}.$$

Furthermore, since $\boldsymbol{0} \prec \boldsymbol{A}$, we get

$$\left(\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{B}^{-\frac{1}{2}}\right)^{-1} = \boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}} \succeq \boldsymbol{I}.$$

By applying Equation (1) another time, we conclude that

$$\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}} \succeq \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{I}\boldsymbol{B}^{-\frac{1}{2}} \Rightarrow \boldsymbol{A}^{-1} \succeq \boldsymbol{B}^{-1}.$$

Exercise 2

For a matrix $\boldsymbol{Z}$ to be the pseudoinverse of a symmetric matrix $\boldsymbol{M}$, you need to show that

1. $\boldsymbol{Z}^\top = \boldsymbol{Z}$.
2. $\boldsymbol{Z}\boldsymbol{v} = \boldsymbol{0}$ for $\boldsymbol{v} \in \ker(\boldsymbol{M})$.
3. $\boldsymbol{M}\boldsymbol{Z}\boldsymbol{v} = \boldsymbol{v}$ for $\boldsymbol{v} \in \ker(\boldsymbol{M})^\perp$.

Prove that if $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are both the pseudo-inverse of symmetric matrix $\boldsymbol{M}$, then $\boldsymbol{Z} = \boldsymbol{Y}$, i.e. the pseudo-inverse is unique.

Solution

Assume $\boldsymbol{Z} \neq \boldsymbol{Y}$. Then there exists some test vector $\boldsymbol{v}$, such that $(\boldsymbol{Z} - \boldsymbol{Y})\boldsymbol{v} \neq \boldsymbol{0}$. Let $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ with $\boldsymbol{v}_1 \in \ker(\boldsymbol{M})^\perp$ and $\boldsymbol{v}_2 \in \ker(\boldsymbol{M})$. But then consider

$$\boldsymbol{M}(\boldsymbol{Z} - \boldsymbol{Y})\boldsymbol{v} = \boldsymbol{M}\boldsymbol{Z}\boldsymbol{v}_1 - \boldsymbol{M}\boldsymbol{Y}\boldsymbol{v}_1 + \boldsymbol{M}\boldsymbol{Z}\boldsymbol{v}_2 - \boldsymbol{M}\boldsymbol{Y}\boldsymbol{v}_2 = \boldsymbol{v}_1 - \boldsymbol{v}_1 = \boldsymbol{0}$$

which implies $(\boldsymbol{Z} - \boldsymbol{Y})\boldsymbol{v} \in \ker(\boldsymbol{M})$. However, this cannot be since $\operatorname{im}(\boldsymbol{Z})^\perp = \operatorname{im}(\boldsymbol{Z}^T)^\perp = \ker(\boldsymbol{Z}) \subseteq \ker(\boldsymbol{M})$ and thus $\operatorname{im}(\boldsymbol{Z} - \boldsymbol{Y}) \perp \ker(\boldsymbol{M})$.

Exercise 3

Let $\boldsymbol{M} = \boldsymbol{X}\boldsymbol{Y}\boldsymbol{X}^\top$ for some $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n \times n}$, where $\boldsymbol{X}$ is invertible and $\boldsymbol{M}$ is symmetric. Furthermore, consider the spectral decomposition of $\boldsymbol{M} = \sum_{i=1}^n \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^\top$. Then, we define $\boldsymbol{\Pi}_{\boldsymbol{M}} = \sum_{i, \lambda_i \neq 0} \boldsymbol{v}_i \boldsymbol{v}_i^\top$. $\boldsymbol{\Pi}_{\boldsymbol{M}}$ is the orthogonal projection onto the image of $\boldsymbol{M}$: It has the property that for $\boldsymbol{v} \in \operatorname{im}(\boldsymbol{M})$, $\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{v}$ and for $\boldsymbol{v} \in \ker(\boldsymbol{M})$, $\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{0}$.

Prove that

$$\boldsymbol{Z} = \boldsymbol{\Pi}_{\boldsymbol{M}}(\boldsymbol{X}^\top)^{-1}\boldsymbol{Y}^+\boldsymbol{X}^{-1}\boldsymbol{\Pi}_{\boldsymbol{M}}$$

is the pseudoinverse of $\boldsymbol{M}$.

Solution

1. Since $(\boldsymbol{v}_i\boldsymbol{v}_i^\top)^\top = \boldsymbol{v}_i\boldsymbol{v}_i^\top$, we have that $\boldsymbol{\Pi}_{\boldsymbol{M}}^\top = \boldsymbol{\Pi}_{\boldsymbol{M}}$. Furthermore, we observe that $(\boldsymbol{X}^\top)^{-1} = (\boldsymbol{X}^{-1})^\top$ and $(\boldsymbol{Y}^+)^\top = \boldsymbol{Y}^+$. Therefore, we get

$$\begin{aligned}
\boldsymbol{Z}^\top &= \left(\boldsymbol{\Pi}_{\boldsymbol{M}}(\boldsymbol{X}^\top)^{-1}\boldsymbol{Y}^+\boldsymbol{X}^{-1}\boldsymbol{\Pi}_{\boldsymbol{M}}\right)^\top \\
&= \boldsymbol{\Pi}_{\boldsymbol{M}}^\top \left(\boldsymbol{X}^{-1}\right)^\top \left(\boldsymbol{Y}^+\right)^\top \left(\left(\boldsymbol{X}^\top\right)^{-1}\right)^\top \boldsymbol{\Pi}_{\boldsymbol{M}}^\top \\
&= \boldsymbol{\Pi}_{\boldsymbol{M}} \left(\boldsymbol{X}^\top\right)^{-1} \boldsymbol{Y}^+\boldsymbol{X}^{-1}\boldsymbol{\Pi}_{\boldsymbol{M}} \\
&= \boldsymbol{Z}.
\end{aligned}$$

2. Consider an arbitrary vector $\boldsymbol{v} \in \ker(\boldsymbol{M})$. We know that $\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{0}$. Therefore, we have

$$\boldsymbol{Z}\boldsymbol{v} = \boldsymbol{\Pi}_{\boldsymbol{M}} \left(\boldsymbol{X}^\top\right)^{-1} \boldsymbol{Y}^+\boldsymbol{X}^{-1}\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{v} = \boldsymbol{0}.$$

3. Let $\boldsymbol{v}$ be an arbitrary vector in $\ker(\boldsymbol{M})^\perp$. We want to prove that $\boldsymbol{M}\boldsymbol{Z}\boldsymbol{v} = \boldsymbol{v}$. We know that $\ker(\boldsymbol{M})^\perp = \operatorname{im}(\boldsymbol{M})$. This implies that $\boldsymbol{v} = \boldsymbol{M}\boldsymbol{z}$ for some vector $\boldsymbol{z}$. Therefore, it suffices to

show that $\boldsymbol{M}\boldsymbol{Z}\boldsymbol{M}\boldsymbol{z} = \boldsymbol{M}\boldsymbol{z}$. We observe that $\boldsymbol{M}\boldsymbol{\Pi}_{\boldsymbol{M}} = \boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{M} = \boldsymbol{M}$. Thus, we have that

$$\begin{aligned}
\boldsymbol{M}\boldsymbol{Z}\boldsymbol{M}\boldsymbol{z} &= (\boldsymbol{M}\boldsymbol{\Pi}_{\boldsymbol{M}}) \left(\boldsymbol{X}^\top\right)^{-1} \boldsymbol{Y}^+\boldsymbol{X}^{-1} (\boldsymbol{\Pi}_{\boldsymbol{M}}\boldsymbol{M})\, \boldsymbol{z} \\
&= \boldsymbol{M} \left(\boldsymbol{X}^\top\right)^{-1} \boldsymbol{Y}^+\boldsymbol{X}^{-1}\boldsymbol{M}\boldsymbol{z} \\
&= \boldsymbol{X}\,\boldsymbol{Y}\boldsymbol{X}^\top \left(\boldsymbol{X}^\top\right)^{-1} \boldsymbol{Y}^+\boldsymbol{X}^{-1}\boldsymbol{X}\,\boldsymbol{Y}\boldsymbol{X}^\top\boldsymbol{z} \\
&= \boldsymbol{X}\,\boldsymbol{Y}\,\boldsymbol{Y}^+\boldsymbol{Y}\boldsymbol{X}^\top\boldsymbol{z} \\
&= \boldsymbol{X}\,\boldsymbol{Y}\boldsymbol{X}^\top\boldsymbol{z} \\
&= \boldsymbol{M}\boldsymbol{z}.
\end{aligned}$$

Exercise 4

Suppose that a weighted graph G is a ϕ-expander, with Laplacian $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$.

1. Prove that for any $\boldsymbol{z} \perp \mathbf{1}$,

   $$\boldsymbol{z}^\top\boldsymbol{L}^\dagger\boldsymbol{z} \preceq 2\phi^{-2}\boldsymbol{z}^\top\boldsymbol{D}^{-1}\boldsymbol{z}.$$

   Hint: Use the result from Exercise 3 in Problem Set 5.

2. Use the statement above to give an upper bound on the effective resistance between any two vertices u, v of G.

Solution.

1. Note that the null space of $\boldsymbol{D}^{-1/2}\boldsymbol{L}\boldsymbol{D}^{-1/2}$ is spanned by $\boldsymbol{D}^{1/2}\mathbf{1}$, as $\mathbf{1}$ spans the null space of $\boldsymbol{L}$. Cheeger's inequality gives that for $\boldsymbol{y}\perp\boldsymbol{D}^{1/2}\mathbf{1}$,

   $$\boldsymbol{y}^\top\boldsymbol{D}^{-1/2}\boldsymbol{L}\boldsymbol{D}^{-1/2}\boldsymbol{y} \geq 0.5\phi^2\boldsymbol{y}^\top\boldsymbol{y}.$$

   We let $\mathbf{Q}$ denote the projection orthogonal to $\boldsymbol{D}^{1/2}\mathbf{1}$. We can then equivalently write

   $$\boldsymbol{D}^{-1/2}\boldsymbol{L}\boldsymbol{D}^{-1/2} \succeq 0.5\phi^2\mathbf{Q}.$$

   From this we conclude that,

   $$(\boldsymbol{D}^{-1/2}\boldsymbol{L}\boldsymbol{D}^{-1/2})^\dagger \preceq 2\phi^{-2}\mathbf{Q}^\dagger$$

   as $\boldsymbol{A} \succeq \boldsymbol{B}$ implies $\boldsymbol{A}^\dagger \preceq \boldsymbol{B}^\dagger$ when $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same null space. Hence by using Exercise 12 from problem set 2 and $\mathbf{Q} = \mathbf{Q}^\dagger$, we then get

   $$\mathbf{Q}\boldsymbol{D}^{1/2}\boldsymbol{L}^\dagger\boldsymbol{D}^{1/2}\mathbf{Q} \preceq 2\phi^{-2}\mathbf{Q}.$$

   This we can rewrite as for all $\boldsymbol{y}\perp\boldsymbol{D}^{1/2}\mathbf{1}$.

   $$\boldsymbol{y}^\top\boldsymbol{D}^{1/2}\boldsymbol{L}^\dagger\boldsymbol{D}^{1/2}\boldsymbol{y} \leq 2\phi^{-2}\boldsymbol{y}^\top\boldsymbol{y}.$$

   Substituting $\boldsymbol{z} = \boldsymbol{D}^{1/2}\boldsymbol{y}$ changes the constraint to $\boldsymbol{D}^{-1/2}\boldsymbol{z}\perp\boldsymbol{D}^{1/2}\mathbf{1}$ i.e. $\boldsymbol{z}\perp\mathbf{1}$. Thus we have that for all $\boldsymbol{z}\perp\mathbf{1}$.

   $$\boldsymbol{z}^\top\boldsymbol{L}^\dagger\boldsymbol{z} \preceq 2\phi^{-2}\boldsymbol{z}^\top\boldsymbol{D}^{-1}\boldsymbol{z}.$$

2. If $\boldsymbol{z} = \boldsymbol{e}_u - \boldsymbol{e}_v$, then $\boldsymbol{z} \perp \mathbf{1}$, so

$$\boldsymbol{z}^\top \boldsymbol{L}^\dagger \boldsymbol{z} \leq 2\phi^{-2} \boldsymbol{z}^\top \boldsymbol{D}^{-1} \boldsymbol{z} = 2\phi^{-2} \left(\frac{1}{\boldsymbol{d}(u)} + \frac{1}{\boldsymbol{d}(v)} \right)$$