Advanced Graph Algorithms and Optimization Spring 2023

Spectral Graph Theory

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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all. This exercise sheet has exercises related to week 6. We encourage you to start early so you have time to go through everything.

To get feedback, you must hand in your solutions by 23:59 on April 7. Both hand-written and LaTeX solutions are acceptable, but we will only attempt to read legible text.

Notation

Througout the following exercises, we will use the following notation:

- S^n is the set of symmetric real matrices $n \times n$ matrices.
- S^n_+ is the set of positive semi-definite $n \times n$ matrices.
- S^n_{++} is the set of positive definite $n \times n$ matrices.

Whenever we say a matrix is positive semi-definite or positive definite, we require it to be real and symmetric.

Exercise 1

In this exercise, we want you to complete the proof of Theorem 9.3.3 in Chapter 9. Refer to the lectures notes for definitions of the terms used here.

1. Prove that Equation (9.4) is satisfied, i.e. that for all edges $e \in E$ we have $\|\boldsymbol{X}_e\| \leq \frac{1}{\alpha}$.
2. Prove that Equation (9.5) is satisfied, i.e. that $\left\|\sum_e \mathbb{E}\left[\boldsymbol{X}_e^2\right]\right\| \leq \frac{1}{\alpha}$.
3. Explain how we can use a scalar Chernoff bound to prove that $\left|\tilde{E}\right| \leq O(\epsilon^{-2}\log(n/\delta)n)$ with probability at least $1-\delta/2$. You may pick any constant that suits you to establish the $O(\cdot)$ bound.

Exercise 2

Consider $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{v}, \boldsymbol{u} \in \mathbb{R}^n$.

1. Assume that $\boldsymbol{I} + \boldsymbol{u}\boldsymbol{v}^\top$ is invertible. Determine c such that

$$\left(\boldsymbol{I} + \boldsymbol{u}\boldsymbol{v}^\top\right)^{-1} = \boldsymbol{I} - \frac{\boldsymbol{u}\boldsymbol{v}^\top}{c}.$$

2. Assume that both $\boldsymbol{A}$ and $\boldsymbol{A}+\boldsymbol{u}\boldsymbol{v}^\top$ are invertible. Prove that

$$\left(\boldsymbol{A}+\boldsymbol{u}\boldsymbol{v}^\top\right)^{-1} = \boldsymbol{A}^{-1} - \frac{\boldsymbol{A}^{-1}\boldsymbol{u}\boldsymbol{v}^\top\boldsymbol{A}^{-1}}{1+\boldsymbol{v}^\top\boldsymbol{A}^{-1}\boldsymbol{u}}.$$

Hint: You might use that $(\boldsymbol{B}\boldsymbol{C})^{-1} = \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}$ *for two invertible matrices* $\boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{n\times n}$.

Exercise 3

Consider a matrix function $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$. For $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n\times n}$, we define

$$Df\left(\boldsymbol{X}\right)\left[\boldsymbol{Y}\right] = \frac{\partial}{\partial t}\Big|_{t=0} f\left(\boldsymbol{X}+t\,\boldsymbol{Y}\right).$$

Remark. *Note that if we think of* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *each as a vector of numbers, then this is the (matrix-valued) directional derivative of* f *at* $\boldsymbol{X}$ *in the direction of* $\boldsymbol{Y}$.

Consider $f(\boldsymbol{X}) = \boldsymbol{X}^{-1}$ for an invertible matrix $\boldsymbol{X} \in \mathbb{R}^{n\times n}$. Prove that

$$Df(\boldsymbol{X})[\boldsymbol{Y}] = -\boldsymbol{X}^{-1}\boldsymbol{Y}\boldsymbol{X}^{-1}.$$

Hint: You might need to use Exercise 3.

Exercise 4

1. Consider $\boldsymbol{A} \in S^n_{++}$ and matrix $\boldsymbol{\Delta} \in S^n_{+}$. Prove that $(\boldsymbol{A}+\boldsymbol{\Delta})^{-1} \preceq \boldsymbol{A}^{-1}$.

2. Let T be a convex set. We say that a function $f : T \to \mathbb{R}^{n\times n}$, is operator convex if for any two matrices $\boldsymbol{A}, \boldsymbol{B} \in T$ and any $\theta \in [0,1]$

$$f\left(\theta\boldsymbol{X} + (1-\theta)\ \boldsymbol{Y}\right) \preceq \theta f\left(\boldsymbol{X}\right) + (1-\theta)\, f\left(\boldsymbol{Y}\right).$$

   Prove that $f(\boldsymbol{X}) = \boldsymbol{X}^{-1}$ is operator convex over the set $T = S^n_{++}$.

   Hint: You could first show that operator convexity is implied by the second directional derivative $D^2 f(\boldsymbol{X})[\boldsymbol{Y}, \boldsymbol{Y}]$ *being positive semi-definite for all* $\boldsymbol{Y} \in S^n$ *and* $\boldsymbol{X} \in S^n_{++}$.