Advanced Graph Algorithms and Optimization Spring 2024

(Mostly) Convex Duality

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The exercises for this week will not count toward your grade, but you are highly encouraged to solve them all. This exercise sheet has exercises related to week 8. We encourage you to start early so you have time to go through everything.

To get feedback, you must hand in your solutions by 23:59 on May 9. Both hand-written and LaTeX solutions are acceptable, but we will only attempt to read legible text.

Exercise 1: Different Duals

Let $G = (V, E)$ be a directed graph with capacities $\boldsymbol{c} \in \mathbb{R}^E \geq \boldsymbol{0}$, and edge-vertex incidence matrix $\boldsymbol{B}$, and consider a demand vector $\boldsymbol{d} \in \mathbb{R}^V$ with $\boldsymbol{d} \perp \boldsymbol{1}$.

Let us try to minimize $\left\| \boldsymbol{C}^{-1}\boldsymbol{f} \right\|_\infty$, where $\boldsymbol{C} = \text{diag}_{e \in E}\, \boldsymbol{c}(e)$. This leads to an optimization problem

$$\begin{aligned} \min_{\boldsymbol{f} \in \mathbb{R}^E_{\geq 0}} & \left\| \boldsymbol{C}^{-1}\boldsymbol{f} \right\|_\infty \\ \text{s.t. } & \boldsymbol{B}\boldsymbol{f} = \boldsymbol{d} \end{aligned}$$

- Compute the dual of this problem. Note that we encoded the "constraint" $\boldsymbol{f} \geq \boldsymbol{0}$ in the domain of the variable $\boldsymbol{f}$. This means that there will *not* be a dual variable associated with this constraint.
- Does Strong Duality hold for this pair of primal and dual problems?

Consider also a variant of this problem, where we instead treat $\boldsymbol{f} \geq \boldsymbol{0}$ as an explicit constraint:

$$\begin{aligned} \min_{\boldsymbol{f} \in \mathbb{R}^E} & \left\| \boldsymbol{C}^{-1}\boldsymbol{f} \right\|_\infty \\ \text{s.t. } & \boldsymbol{B}\boldsymbol{f} = \boldsymbol{d} \\ & \boldsymbol{f} \geq \boldsymbol{0} \end{aligned}$$

This problem has a different dual problem because we made $\boldsymbol{f} \geq \boldsymbol{0}$ an explicit constraint.

- Compute the dual of this variant of the problem.
- Explain how, given an optimal solution to the first dual problem, we can compute an optimal solution to the second dual problem.

Exercise 2: A "Broken" Dual

Consider the following optimization problem

$$\min_{x \in \mathbb{R}, y \in \mathbb{R}_{>0}} e^{-x}$$
$$\text{s.t. } x^2/y \leq 0.$$

- Compute the dual program.
- What is the optimal value of the primal program? And of the dual program? Does strong duality hold? Does Slater's condition hold?

Exercise 3: Norms and a Lagrangian

Suppose $1 < q < p < \infty$. In this exercise, we want to prove that for $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$\|\boldsymbol{x}\|_q \leq n^{1/q - 1/p} \|\boldsymbol{x}\|_p . \tag{1}$$

Consider the following optimization problem:

$$\max_{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_p^p \leq 1} \|\boldsymbol{x}\|_q^q$$

- Is this a convex optimization problem?
- Informally, explain why, at any maximizing $\boldsymbol{x}$ for the above problem, there should exist a $\lambda > 0$ such that
$$\boldsymbol{\nabla}_{\boldsymbol{x}} \left(\|\boldsymbol{x}\|_q^q - \lambda \|\boldsymbol{x}\|_p^p \right) = \boldsymbol{0}.$$
- Use the existence of such a λ to prove Equation (1).

Exercise 4: Flows and Voltages and Other Powers

Consider a connected undirected graph $G = (V, E)$ with resistances $\boldsymbol{r} \in \mathbb{R}^E$ and edge-vertex incidence matrix $\boldsymbol{B}$, and a demand vector $\boldsymbol{d} \in \mathbb{R}^V$ with $\boldsymbol{d} \perp \boldsymbol{1}$.

Given some $p > 1$, we'll look at the flow optimization problem

$$\min_{\boldsymbol{f} \in \mathbb{R}^E} \sum_e \boldsymbol{r}(e) \frac{1}{p} |\boldsymbol{f}(e)|^p$$
$$\text{s.t. } \boldsymbol{B}\boldsymbol{f} = \boldsymbol{d}.$$

- Is the above optimization problem convex?
- Does Slater's condition hold for the problem?

- What is the dual problem for the problem? Define $q > 0$ to be the number such that $1 = \frac{1}{q} + \frac{1}{p}$. Try to find a clean expression of the dual. Writing the expression in terms of q instead of p will simplify it.

Suppose our instance of the optimization problem as an optimal flow solution $\boldsymbol{f}^*$. Let $\alpha = \sum_e \boldsymbol{r}(e) \frac{1}{p} |\boldsymbol{f}^*(e)|^p$ be the optimal problem value. Suppose that for some particular edge $\hat{e}$ we have

$$\gamma = \frac{\boldsymbol{r}(\hat{e}) \frac{1}{p} |\boldsymbol{f}^*(\hat{e})|^p}{\alpha}.$$

Now, consider a modified instance with resistances $\tilde{\boldsymbol{r}} \in \mathbb{R}^E$ given by

$$\tilde{\boldsymbol{r}}(e) = \begin{cases} \boldsymbol{r}(e) & \text{for } e \neq \hat{e} \\ 2^{p-1} \boldsymbol{r}(e) & \text{for } e = \hat{e} \end{cases}$$

That is, we increase the resistance on edge $\hat{e}$ by a factor 2^{p-1}. Let $\tilde{\alpha}$ denote optimal value of program with the new resistances $\tilde{\boldsymbol{r}}$. Prove that

$$\tilde{\alpha} \geq \left(1 + \frac{p-1}{2} \gamma\right) \alpha.$$

Hint: use the dual problem!