

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

Group A: Wed 14–16 CAB G 56

Group B: Wed 14–16 CAB G 57

Group C: Wed 16–18 CAB G 56

Group D: Wed 16–18 CAB G 57

- This is a theory course, which means: if an exercise does not explicitly say “you do not need to prove your answer”, then a formal proof is **always** required.

The following exercises will be discussed in the exercise classes on October 11, 2023. Please hand in your solutions via Moodle, no later than 2 pm at October 10.

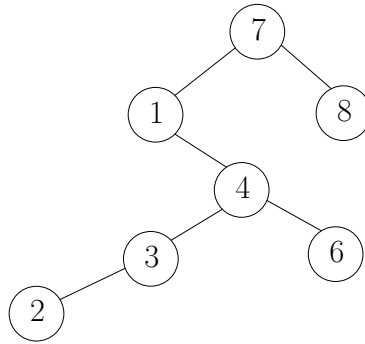
Exercise 1

Let $n \in \mathbf{N}$. Show that the expected number of nodes of depth $n - 1$ in a random search tree for n keys is $\frac{2^{n-1}}{n!}$. What is the probability that there is a node of depth $n - 1$?

Exercise 2

Let S_n denote the number of keys that are descendants of the smallest key. For example, in the tree below, $S_n = 5$, because the elements 1, 2, 3, 4, 6 are descendants of 1.

Compute $E[S_n]$.



Exercise 3

Determine closed forms for the following recursively defined series:

(1) For $n \in \mathbf{N}$,

$$a_n = \begin{cases} 1, & \text{if } n = 1, \text{ and} \\ 1 + \frac{1}{n} \sum_{i=1}^{n-1} a_i, & \text{otherwise.} \end{cases}$$

(2) For $n \in \mathbf{N}$,

$$b_n = \begin{cases} 1, & \text{if } n = 1, \text{ and} \\ 2 + \sum_{i=1}^{n-1} b_i, & \text{otherwise.} \end{cases}$$

(3) For $n \in \mathbf{N}_0$,

$$c_n = \begin{cases} 0, & \text{if } n = 0, \text{ and} \\ n - 1 + \sum_{i=1}^n \frac{c_{i-1} + c_{n-i}}{2}, & \text{otherwise.} \end{cases}$$

(4) For $n \in \mathbf{N}_0$,

$$d_n = \begin{cases} 0, & \text{if } n = 0, \text{ and} \\ 1 + 2 \sum_{i=0}^{n-1} (-1)^{n-i} d_i, & \text{otherwise.} \end{cases}$$

(5) For $n \in \mathbf{N}_0$,

$$e_n = \begin{cases} 1, & \text{if } n = 0, \text{ and} \\ 1 + n e_{n-1}, & \text{otherwise.} \end{cases}$$

Exercise 4

Let $n \in \mathbf{N}$. Determine the expected number of leaves in a random search tree for n keys.

Exercise 5

Consider the process of inserting the keys $\{1, 2, \dots, n\}$ into an empty treap in the order $(1, 2, \dots, n)$.

- (a) During this process, what is the expected number of changes of the root of the treap? (We also count the very first insertion as a change of the root.)
- (b) For a given key i : What is the probability that i occurs as the right child of the root (after an insertion, i.e., with necessary rotations completed) in the process?
- (c) What is the expected number of elements that occur as the left child of the root (after an insertion, i.e., with necessary rotations completed) in the process?

Exercise 6

Let $i, j, n \in \mathbf{N}$, $i < j \leq n$. What is the probability that the randomized procedure `quicksort()` applied to a set of n numbers compares the element of rank i with the element of rank j ?

Hint: If you are stuck, you might want to read section 2.4 of the script.