ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Bernd Gärtner, Rasmus Kyng, Angelika Steger, David Steurer, Emo Welzl

Algorithms, Probability, and Computing — Exercises KW47 — HS23

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A: Wed 14–16 CAB G 56

  Group B: Wed 14–16 CAB G 57

  Group C: Wed 16–18 CAB G 56

  Group D: Wed 16–18 CAB G 57

- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

The following exercises will be discussed in the exercise classes on November 22, 2023. Please hand in your solutions via Moodle, no later than 2 pm at November 21.

Exercise 1

Show that every feasible point of the Tight Spanning Tree LP is feasible in the Loose Spanning Tree LP – without using theorem 4.11.

Exercise 2

Consider the following linear program, almost the Tight Spanning Tree LP, it seems:

Some LP for graph $G = (V, E), c \in \mathbf{R}^E$

$$\begin{array}{lrcll} \min & c^T x & & & \\ \text{subject to} & \sum_{e \in E} x_e & = & n & \\ & \sum_{e \in E \cap \binom{S}{2}} x_e & \leq & |S| - 1, & \text{for all } S \subseteq V,\ \emptyset \neq S \neq V, \text{ and} \\ & 1 \geq x_e & \geq & 0, & \text{for all } e \in E. \end{array}$$

What are the edge sets corresponding to vectors $x \in \{0, 1\}^E$ feasible in Some LP?

Exercise 3

Let $D = (V, A)$ be a directed graph and let $s, t \in V$. To any vertex set $S \subseteq V$ we associate a *cut* $C(S) \subseteq A$ that consists of all arcs between S and $V \setminus S$. We say that $C(S)$ is an *s-t cut* if $s \in S$ and $t \notin S$. We say that $C(S)$ is a *strong* s-t *cut* if it is an s-t cut and if all edges in $C(S)$ are directed away from $V \setminus S$. See Figure 1 for an example.

In this exercise we will prove the following lemma and see that it is a special case of the Farkas lemma we have seen in the lecture. Informally, it says that there is a simple certificate for both proving and disproving the existence of a directed s-t path in D.

Lemma 1 (Farkas lemma for s-t-paths). *Exactly one of the following two statements holds for any directed graph $D = (V, A)$ and for any two vertices $s, t \in V$.*

i) *There exists a directed s-t path.*

ii) *There exists a strong s-t cut.*

For every vertex $v \in V$ let $\delta(v)^+ \subseteq A$ denote the arcs that are outgoing from v and let $\delta(v)^- \subseteq A$ denote the arcs that are incoming to v.

(a) Show that there is a directed s-t path in D if and only if the following system of equations and inequalities has a solution over the real valued variables $\{x_e \mid e \in A\}$.

$$\forall v \in V: \quad \sum_{e \in \delta(v)^+} x_e - \sum_{e \in \delta(v)^-} x_e = \begin{cases} 0 & \text{if } v \in V \setminus \{s,t\} \\ 1 & \text{if } v = s \\ -1 & \text{if } v = t \end{cases}$$

$$\forall e \in A: \quad x_e \geq 0$$

(b) Prove Lemma 1 by applying some version of Farkas lemma to the system in (a).

(c) Prove Lemma 1 directly without using (a) or Farkas lemma.

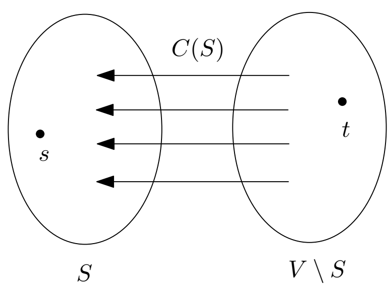

Figure 1: An illustrative example of a strong s-t cut. The cut $C(S)$ is a strong s-t cut because all edges in $C(S)$ are directed away from $V \setminus S$.

Exercise 4

Suppose we are running the checking algorithm for matrices over $GF(2)$, i.e. numbers are $\{0, 1\}$ with addition and multiplication $\mod 2$. Show that in one iteration the success probability of detecting an error in the supposed product matrix C is exactly $\frac{1}{2}$, in case matrix C is wrong in exactly one row.

Exercise 5

For $n \in \mathbf{N}$, let $A \in \mathbf{R}^{n \times n}$ be a non-zero matrix (i.e. not all entries are 0) and let x be a vector u.a.r. from $\{-1, 0, +1\}^n$. Show that the probability that the vector Ax is non-zero is at least $2/3$.

Exercise 6

Given a finite set S of rational numbers and positive integers d and n, $d \leq |S|$, find a polynomial $p(x_1, x_2, \ldots, x_n)$ of degree d for which the Schwartz–Zippel theorem is tight. That is, the number of n-tuples $(r_1, \ldots, r_n) \in S^n$ with $p(r_1, \ldots, r_n) = 0$ is $d|S|^{n-1}$.