Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Mohsen Ghaffari, Angelika Steger, David Steurer, Emo Welzl, Peter Widmayer

Algorithms, Probability, and Computing — Final Exam — HS17

Candidate

First name: ....

Last name: ....

Student ID (Legi) Nr.: ....

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions

1. Check your exam documents for completeness (pages numbered until p. 11, last page empty).
2. You can solve the exercises in any order. You should not be worried if you cannot solve all exercises! Not all points are required to get the best grade.
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints will not be accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. No auxiliary material allowed. All electronic devices must be turned off and are not allowed to be on your desk. We will write the current time on the blackboard every 15 minutes.
6. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. Unless stated otherwise, no points will be awarded for unfounded or incomprehensible solutions. You can write your solutions in English or German.**
9. You may use anything that has been introduced and proved in the lecture or in the exercise sessions without reproving it – unless we explicitly ask you to reproduce a proof. However, if you need something *different* than what we have shown, you must write a new proof or at least list all necessary changes.
10. Write your student-ID (**Legi-number**) on **all** sheets.

Good luck!

	achieved points (maximum)
1	(5)
2	(10)
3	(10)
4	(10)
5	(10)
6	(10)
7	(10)
8	(10)
Σ	(75)

Exercise 1: Randomized algebraic algorithms (5 points)

(a) **(2 points)** Let $S = \{1, 2, \dots, 5\}$. Define a nonzero polynomial $p(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3]$ that has degree 3 and that vanishes for the maximum number of points in S^3. In other words, the quantity $|\{(s_1, s_2, s_3) \in S^3 \mid p(s_1, s_2, s_3) = 0\}|$ should be as large as possible. Remember to argue why your polynomial achieves the optimum.

(b) **(3 points)** Let G be a bipartite graph. Describe a polynomial time deterministic algorithm that decides whether G has an odd number of perfect matchings.

Exercise 2: A recurrence and a search tree (10 points)

(a) **(7 points)** Solve the following recurrence for all positive integers $n \in \mathbb{N}$.

$$a_n = \begin{cases} 4 & \text{if } n = 1 \\ n - 1 + \frac{1}{n} \sum_{i=1}^{n-1} a_i & \text{if } n > 1 \end{cases}$$

(b) **(3 points)** Let T be a random search tree over the keys $[n]$. Fix two keys $k, \ell \in [n]$ with $k \neq \ell$ and let $d(k, \ell)$ denote the length of the path connecting the nodes with keys k and ℓ in T. For example, if k is a parent of ℓ in T then $d(k, \ell) = 1$ and if k and ℓ have a common parent, then $d(k, \ell) = 2$.

For fixed keys k and ℓ, express the random variable $d(k, \ell)$ as an algebraic expression in terms of the ancestor indicator random variables A_j^i where $i, j \in [n]$. You don't need to simplify your expression but you should justify your answer. Recall that $A_j^i = [i \text{ is an ancestor of } j]$.

Exercise 3: Deterministic MIS in the LOCAL model (10 points)

In this question we consider the LOCAL model of computation. We assume that the input graph consists of n vertices with labels $[n]$ and that the graph has maximum degree Δ. In this exercise we will develop a fast deterministic algorithm for computing a maximal independent set (MIS).

(a) **(3 points)** There exists a deterministic $O(\Delta + \log^* n)$ round algorithm for computing a $(\Delta + 1)$-coloring of a graph. How to use it to get a deterministic algorithm for MIS with the same round complexity? Remember to justify your answer.

(b) **(7 points)** The algorithm in (a) is bad if Δ is large. Develop a deterministic algorithm for MIS that runs in $O(\sqrt{n})$ rounds. HINT: Try to reduce the maximum degree fast and then use (a).

Exercise 4: Minimum cut (10 points)

Let G be a connected multigraph with $n \geq 3$ vertices and that has a minimum cut of size $k = \mu(G)$. The probabilities $\Pr[\mu(G) \neq \mu(G \setminus e)]$ below are always with respect to a uniformly at random chosen edge e from the graph G.

(a) **(2 points)** Assume $k = 1$. Prove that $\Pr[\mu(G) \neq \mu(G \setminus e)] \leq \frac{1}{n}$.

(b) **(1 point)** Assume $k = 1$. Show that the basic random edge contraction algorithm $\textsc{BasicMinCut}(G)$ outputs the correct number $\mu(G) = 1$ with probability at least $\frac{2}{n}$.

(c) **(2 points)** For any k show that if G has at least 3 vertices with degree k, then $\Pr[\mu(G) \neq \mu(G \setminus e)] = 0$.

(d) **(5 points)** For any k, show that $\Pr[\mu(G) \neq \mu(G \setminus e)] \leq \frac{2k}{n(k+1)-t}$ where t is a constant. For full points prove that the bound holds for some $t \leq 1$.

Exercise 5: Linear programming (10 points)

(a) **(2 points)** Define what a basic feasible solution of a linear program is.

(b) **(8 points)** Recall that Farkas lemma says that a system $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ of linear inequalities has no solution if and only if there exists a vector $\mathbf{y} \geq 0$ such that $\mathbf{A}^\mathsf{T}\mathbf{y} = \mathbf{0}$ and $\mathbf{b}^\mathsf{T}\mathbf{y} < 0$. Using Farkas lemma, prove Lemma 1 stated below.

Lemma 1 (A variant of Farkas lemma). *A system* $\mathbf{B}\mathbf{x} \leq \mathbf{c}$ *of linear inequalities has no* nonnegative *solution if and only if there exists a vector* $\mathbf{z} \geq 0$ *such that* $\mathbf{B}^\mathsf{T}\mathbf{z} \geq \mathbf{0}$ *and* $\mathbf{c}^\mathsf{T}\mathbf{z} < 0$.

Exercise 6: Mystery LP (10 points)

(a) **(3 points)** Give an example of a connected graph with minimum degree 2 for which the Mystery LP (defined below) is infeasible.

(b) **(1 point)** What do the integer solutions of the Mystery LP correspond to in G?

(c) **(6 points)** Let G be bipartite. Show that if the Mystery LP of G is feasible, then there is also a feasible integer solution.

Mystery LP for a graph $G = (V, E), c \in \mathbf{R}^E$

$$
\begin{array}{lrcll}
\min & c^T x & & & \\
\text{subject to} & \sum_{e \in \delta(v)} x_e & = & 2\,, & \text{for all } v \in V, \\
 & 0 \le x_e & \le & 1\,, & \text{for all } e \in E.
\end{array}
$$

Exercise 7: Applying Chernoff bound (10 points)

You are running a company that owns n servers. For some constant $\epsilon > 0$ you have been given $n^{1+\epsilon}$ computation jobs that you need to assign to the servers. You decide to distribute each job to a uniformly at random chosen server.

Find a function $M = M(n)$ so that the probability of any server receiving more than M jobs is at most $O(\frac{1}{n^2})$. You should aim for a nontrivial function $M(n)$. Below you are given the Chernoff bound which you may find useful.

Chernoff Bound. *Suppose $X_1, X_2, \ldots, X_\ell$ are independent random variables taking values in $\{0, 1\}$. Let $X = \sum_{i=1}^{\ell} X_i$ denote their sum and let $\mu = \mathbb{E}[X]$ denote the sum's expected value. For any $0 < \delta \leq 1$, we have*

$$\Pr[X \notin [\mu(1-\delta), \mu(1+\delta)]] \leq 2e^{-\delta^2\mu/3}.$$

Exercise 8: A simultaneously bisecting line (10 points)

Let ℓ be a line and let P be a finite point set, both in the plane. We say that ℓ *bisects* P if either of the two open halfplanes bounded by ℓ contains at most $|P|/2$ points of P.

(a) **(2 points)** Given that a non-vertical line ℓ bisects P, what property does the dual point ℓ^* satisfy with respect to the set of dual lines $P^* = \{p^* \mid p \in P\}$?

(b) **(2 points)** Let $|P|$ be even and let $p \in P$ be some point. Show that a bisecting line for $P \setminus \{p\}$ is also a bisecting line for P.

(c) **(6 points)** Let P and Q be two finite point sets in the plane. Assume that no three points of $P \cup Q$ lie on a common line and that no two points of $P \cup Q$ have the same x-coordinate. Prove that there always exists a line that simultaneously bisects both P and Q. See Figure 1 for an example of a simultaneously bisecting line. HINT: Consider the levels of the dual line arrangements of P and Q,

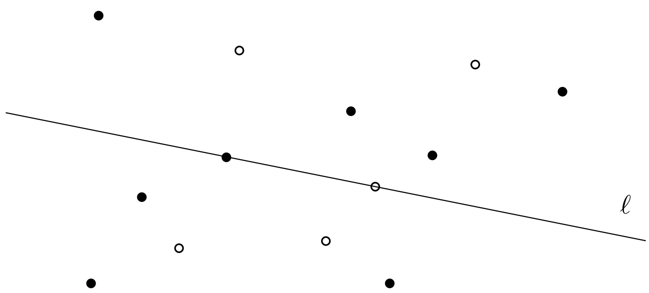

Figure 1: An example for question (c). The line ℓ simultaneously bisects both the solid and the hollow points.