Institute of Theoretical Computer Science
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November 15, 2010

Algorithms, Probability, and Computing *Fall 2010*
Mid-Term Exam

Candidate:

First name: ....

Last name: ....

Student ID (Legi) Nr.: ....

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions:

1. You can solve the 7 exercises in any order. **You should not be worried if you cannot solve all the exercises!** Not all points are necessary in order to get the best grade.
2. Check your exam documents for completeness (1 cover sheet and 3 sheets containing 7 exercises).
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints are not accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. No auxiliary material allowed.
6. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. No points will be awarded for unfounded or incomprehensible solutions (except in the multiple-choice parts). You can write your solution in English or German.**
9. Make sure to write your student-ID (**Legi-number**) on **all** the sheets (and your name only on this cover sheet).

Good luck!

	achieved points (maximum)	reviewer's signature
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6	(10)	
7	(10)	
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Exercise 1

Consider the following 5 claims and mark the corresponding boxes. Grading: 1 point for each correct marking, 2 points for a correct marking with a correct short justification (or reference), and -1 point for a wrongly marked box (you will receive non-negative total points in any case).

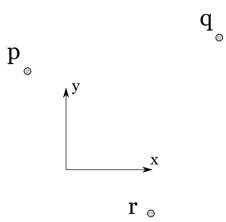
(a) Let T be a random search tree on n nodes. For any given node $k \in \{1..n\}$, consider the operation `swap`(k) which exchanges the (potentially empty) left and right subtrees of k (and relabels the nodes so that we again have a tree in $\mathcal{B}_{\{1..n\}}$).

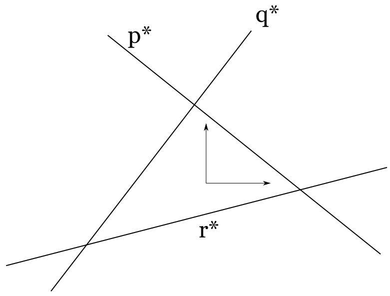
Suppose a tree $T' \in \mathcal{B}_{\{1..n\}}$ can be obtained from the tree $T \in \mathcal{B}_{\{1..n\}}$ by some sequence of `swap`-operations. Then T and T' do have the same probability as random search trees.

[ ] True [ ] False

Justification: ....

(b) Consider three points p, q and r in the plane together with their standard dual lines as follows:





The set of duals of all lines that go through r, and pass between p and q, is convex.

[ ] False [ ] True

Justification: ....

(c) A set of n points in the plane can be pre-processed in time $\mathcal{O}(n^3)$ and linear storage such that the set of points below a query line can be reported in time $\mathcal{O}(\log n)$.

[ ] True [ ] False

Justification: ....

(d) Every network with integral capacities has a maximum flow which is integral.

[ ] False [ ] True

Justification: ....

(e) Every network with (non-negative) real capacities has a maximum flow.

[ ] True [ ] False

Justification: ....

Exercise 2

(a) Solve the following recurrence for $n \in \mathbf{N}_0$.

$$a_n = \begin{cases} 1 + 2\sum_{i=1}^{n} a_{i-1} & \text{for } n \geq 1, \\ 7 & \text{for } n = 0. \end{cases}$$

(b) Solve the following recurrence for $n \in \mathbf{N}_0$.

$$b_n = \begin{cases} 1 + 2\sum_{i=1}^{n} (-1)^i b_{i-1} & \text{for } n \geq 1, \\ 7 & \text{for } n = 0. \end{cases}$$

(c) What is the expected length (number of nodes) of a "central path" in a random search tree on n nodes? (Here "central" means we go left from the root, then alternate between going right and left until we reach a leaf or an empty subtree)

Exercise 3

We are given a set P of n points in $\mathbb{R}^2$ such that no two points from P have the same distance. We add the points of P in random order to a set Q (which is initially empty). For $i = 2, \ldots, n$ let $\{q_i, q_i'\}$ denote the point pair in Q of minimal distance, after the i-th insertion of a point:

$$\|q_i - q_i'\| = \min_{\{s,t\} \in \binom{Q}{2}} \|s - t\|$$

For $|Q| < 2$ the pair of minimal distance is not defined.

Compute the expected number of distinct nearest neighbor pairs that appear during the process. In other words: what is $\mathbf{E}[|\{\{q_i, q_i'\} : i = 2, \ldots, n\}|]$?

Exercise 4

Let (V, E) be an undirected graph. We define a function f on all subsets of vertices: $f : 2^V \to \mathbb{R}$, $f(S) := |E(S, V \setminus S)|$ for $S \subseteq V$. In other words if we think of the set S being a cut of the graph, then $f(S)$ is the size of that cut.

(a) Show that the function f is *sub-modular*, meaning that

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B), \qquad \forall A, B \subseteq V.$$

(b) Assume that both A and B are minimum cuts of the graph G, and suppose that $A \cap B \neq \emptyset$ and $A \cup B \neq V$. Prove that in this case, also $A \cap B$ and $A \cup B$ must be minimum cuts of G. You may assume part (a).

(c) Show that every network with unit capacities has a unique smallest minimum s-t-cut. Here smallest means we order the minimum cuts by the size of the set $S \ni s$. You may assume parts (a) and (b).

Exercise 5

(a) Let P be a set of n points in $\mathbb{R}^2$ in general position. Suppose that n_r of the points in P are colored red and n_b are colored blue ($n_r + n_b = n$, and suppose $n_r, n_b > 10$).

A line ℓ is called *nearly separating* for the point set if there are at most 10 red points on one side of ℓ (or on ℓ), and at most 10 blue points on the other side of ℓ (or on ℓ).

Show that after preprocessing with $\mathcal{O}(n^2)$ space, one can perform the following query: For a non-vertical query line ℓ, one can decide whether ℓ is *nearly separating* in time $\mathcal{O}(\log n)$.

(b) Let P be a set of n points in $\mathbb{R}^2$ in general position. For $i \in \mathbf{N}_0$, let C_i be the intersection of all closed halfplanes that contain at least i points in P. A halfplane is meant to be bounded by a line.

Consider the following 2 claims, and mark the corresponding box. Grading: same as in question 1.

(1) A closed halfplane contains all points in P *if and only if* it contains C_n.

[ ] True [ ] False

Justification: ....

....

(2) A closed halfplane contains at least $n-2$ points in P *if and only if* it contains C_{n-2}.

[ ] True [ ] False

Justification: ....

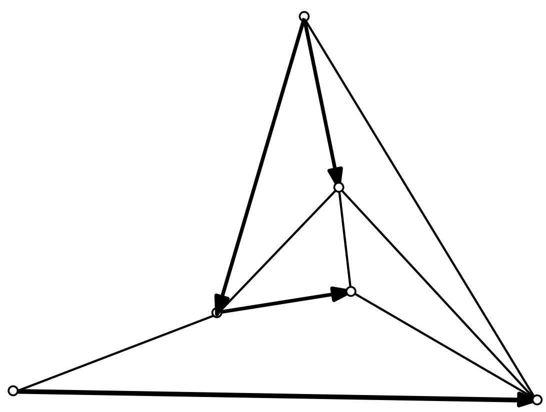
....

Exercise 6

(a) For $n \in \mathbf{N}$, let $A \in \mathbb{R}^{n \times n}$ be a non-zero matrix (i.e. not all entries are 0) and let x be a vector u.a.r. from $\{-1, 0, +1\}^n$. Show that the probability that the vector Ax is non-zero is at least $2/3$.

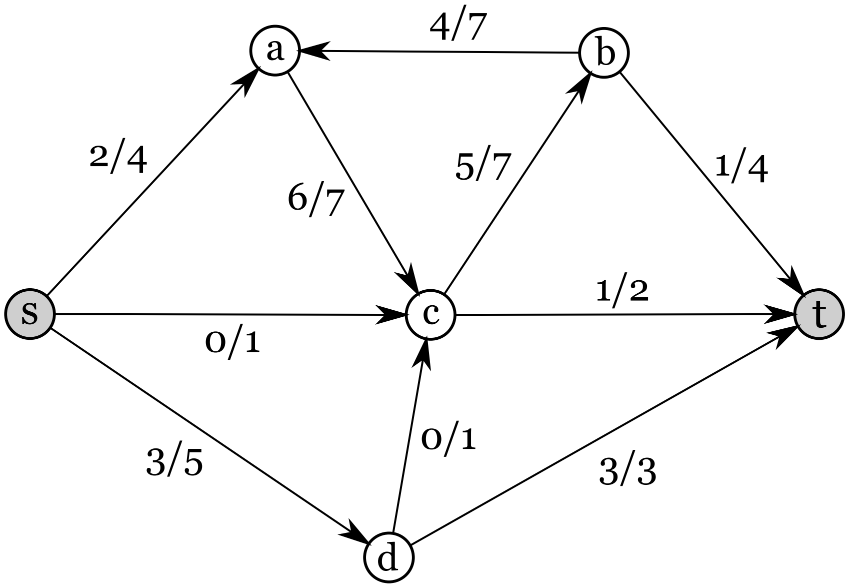
(b) Let G be a graph with no cycle of even length. Show that there is at most one perfect matching in G.

(c) Complete the following partial orientation (four edges are already oriented) to a Pfaffian orientation.



Exercise 7

Consider the following network and its flow. Please note that the indicated numbers 4/7 and so on are not fractions: the 4/7 on the edge from b to a indicates that this edge has capacity 7, and the current flow at this edge has value 4.

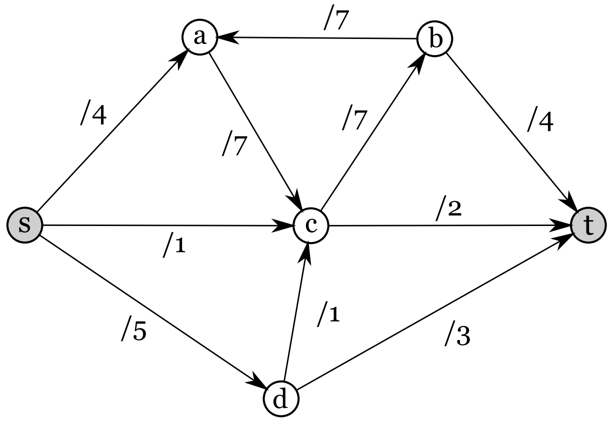


(a) What is the value of the flow?

(b) Draw the residual network.

(c) Simulate the "shortest augmenting path"-variant of the Ford-Fulkerson algorithm on this network: Perform several augemtation steps, starting with the flow drawn above, until you find a maximum flow. In each step, clearly say which augmenting path you are using, and draw the flow after augmenting. For this, please use the copies of the network that are printed on the next page by filling in the flow values.

(d) Give a minimum s-t-cut of this network, i.e. a set S of vertices containing s but not t, minimizing $\text{cap}(S)$.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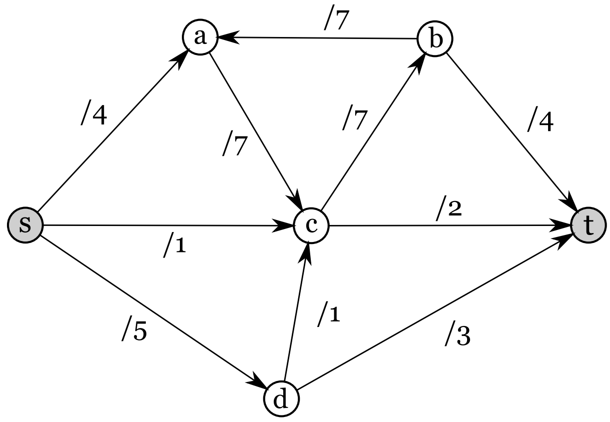
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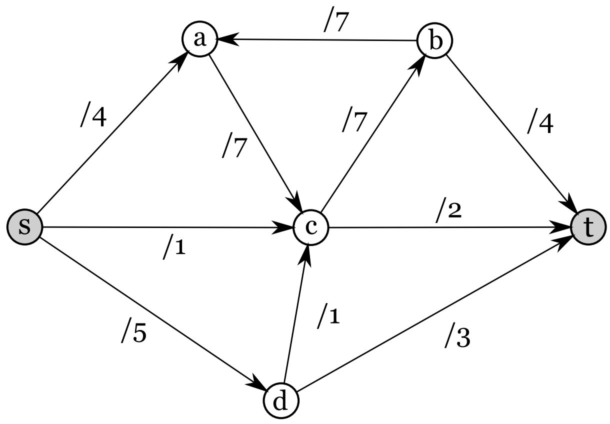
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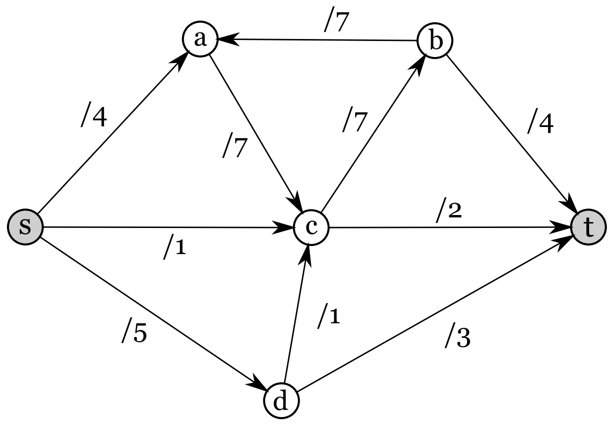
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