Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
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Algorithms, Probability, and Computing — Midterm Exam — HS15

Candidate

First name: ....

Last name: ....

Student ID (Legi) Nr.: ....

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions

1. Check your exam documents for completeness (2 cover pages and 7 pages with 6 exercises).
2. You can solve the six exercises in any order. You should not be worried if you cannot solve all exercises! Not all points are required to get the best grade.
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints will not be accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. No auxiliary material allowed. All electronic devices must be turned off and are not allowed to be on your desk. We will write the current time on the blackboard every 15 minutes.
6. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. Unless stated otherwise, no points will be awarded for unfounded or incomprehensible solutions. You can write your solutions in English or German.**
9. You may use anything that has been introduced and proved in the lecture or in the exercise sessions without reproving it. However, if you need something *different* than what we have shown, you must write a new proof or at least list all necessary changes.
10. Write your student-ID (**Legi-number**) on **all** sheets (and your name only on this cover sheet).

Good luck!

	achieved points (maximum)	reviewer's signature
1	(25)	
2	(15)	
3	(15)	
4	(15)	
5	(15)	
6	(15)	
Σ	(100)	

Exercise 1: Multiple Choice (25 points)

Consider the following five claims/questions and mark the corresponding boxes. Grading: 2 points for a correct marking without a correct justification, 5 points for a correct marking with a correct short justification, and -2 points for a wrongly marked box. You will receive a non-negative total number of points for the whole exercise in any case.

(a) Let F be CNF-formula where each clause has size exactly 4. Then there exists an assignment of F that satisfies at least $15/17$-fraction of the clauses.

[ ] False [ ] True

Justification: ....

....

(b) Let $^\circ : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping that maps points to non-vertical lines, and non-vertical lines to points by

$$\text{point } p = (a, b) \mapsto \text{ line } p^\circ : y = ax + b$$
$$\text{line } \ell : y = ax + b \mapsto \text{ point } \ell^\circ = (a, b).$$

The mapping $^\circ$ is incidence preserving.

[ ] False [ ] True

Justification: ....

....

(c) Let d_n^R be the expected height of a treap on n keys, when the keys are inserted in order $\pi \in_{u.a.r} S_n$. Let d_n^D be the expected height, when the keys are inserted in increasing order. Then $d_n^R = d_n^D$.

[ ] False [ ] True

Justification: ....

....

(d) Given a linear program in equational form, we can always convert it to a linear program of form $\max c^T x$ subject to $Ax \leqslant b$, without adding any additional variables.

[ ] False [ ] True

Justification: ....

....

(e) We are given an array $a_1, \ldots, a_n$ and we are told that the largest element x occurs $\frac{2n}{3}$ times in the array, but we are not told the value of x. There is a constant time randomized algorithm that finds the correct value x with probability at least $\frac{26}{27}$.

[ ] False [ ] True

Justification: ....

....

Exercise 2: Random(ized) Search Trees (5 + 10 points)

(a) For a binary tree on the nodes $[n] = \{1, 2, \ldots, n\}$ we define the indicator variable $X_n^{i,j}$ for $i < j$, $i, j \in [n]$ as follows. $X_n^{i,j} = 1$ iff there exists a node $k \neq i, j$ such that i is in the left subtree of k and j is in the right subtree of k. What is the expected value of $X_n^{i,j}$ in a random binary search tree?

(b) We call a node in a binary tree *right-weighted* if its left subtree has exactly one node and its right subtree has at least one node. What is the expected number of right-weighted nodes in a random binary search tree on n nodes?

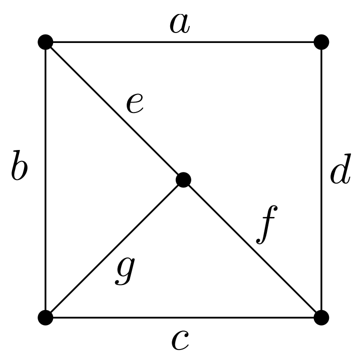
Exercise 3: Point Location ($5 + 10$ points)

(a) Let C be a circle in $\mathbb{R}^2$ with center $(0,0)$ and L a set of n distinct lines through $(0,0)$. Let $l_1, \ldots, l_n$ be a uniformly at random permutation of L and we insert the lines one by one in this order. Then $l_1, \ldots, l_i$ induce an arrangement of open intervals on C. For a query point $q \in C$, compute the exact expected number of distinct intervals q is contained in during the insertion process. The insertion of l_1 is counted as a change and you can assume that q does not lie on any line of L.

(b) Let T be a set of n 2-dimensional triangles in $\mathbb{R}^3$, where $\bigcup_{t \in T} t \subseteq H_0 := \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$, i.e., all triangles lie strictly above the plane defined by $z = 0$. Preprocess T such that in expected time $O(\log n)$ we can decide the following: For a query ray $\overrightarrow{v}$ shooting from the origin into the open halfspace H_0, decide whether $\overrightarrow{v}$ intersects a triangle of T or not. You have to describe how you do the preprocessing but you do not need to give a preprocessing time. You can assume that if $\overrightarrow{v}$ intersects $t \in T$, then the intersection is in the interior of t.

Exercise 4: Minimum Cut (5 + 10 points)

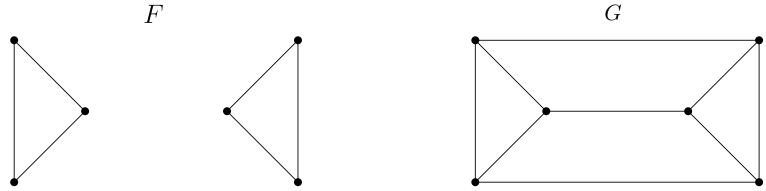
(a) Let G_0 be the graph given below. For a k of your choice give a sequence $e_0, e_1, \dots e_{k-1}$ of edges, such that for $i \in [k]$ $G_i = G_{i-1}/e_{i-1}$ with $e_{i-1} \in G_{i-1}$ and G_k is a graph on two vertices such that $\mu(G_0) = \mu(G_k)$. Do not forget to give a justification.



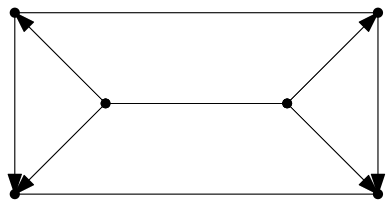
(b) Let $\mathcal{A}$ be some random algorithm that for a given multigraph G_{2n} on $2n$ vertices ($n \geqslant 2$), outputs a multigraph G_n on n vertices, such that with probability at least $\frac{1}{2}$ it holds that $\mu(G_n) = \mu(G_{2n})$ and in any other case $\mu(G_n) > \mu(G_{2n})$. Let $\mathcal{B}$ be the algorithm that takes as in input a multigraph G_{2^k} on 2^k vertices and outputs a multigraph on two vertices as follows: $\mathcal{B}$ runs $\mathcal{A}$ on G_{2^k}. For $k = 2$ output $G_{2^{k-1}}$, otherwise run $\mathcal{B}$ on $G_{2^{k-1}}$. Show that $7 \cdot 2^k \ln k$ runs of $\mathcal{B}$ are sufficient to obtain a multigraph H_2 on two vertices, such that $\mu(G_{2^k}) = \mu(H_2)$ with probability at least $1 - k^{-14}$.

Exercise 5: Randomized Algebraic Algorithms (5 + 10 points)

(a) Recall that every planar graph G has a Pfaffian orientation. Consider the graphs $F = (V_F, E_F)$, $G = (V_G, E_G)$, with $V_F = V_G$, $E_F \subseteq E_G$ as given below. Show that not every Pfaffian orientation $\vec{F} = (V_F, E_{\vec{F}})$ is extendable to a Pfaffian orientation $\vec{G} = (V_G, E_{\vec{G}})$, i.e., a Pfaffian orientation $\vec{G}$ of G such that the restriction of $\vec{G}$ to E_F is $\vec{F}$.



Hint: Use the orientations picture below, but do not forget to argue why you can do this.



(b) Let $p(x_1, x_2) \in \mathbb{F}[x_1, x_2]$ be a nonzero polynomial of degree $d \geqslant 1$ in some field $\mathbb{F}$. Let $S_1, S_2 \subseteq \mathbb{F}$ two finite sets. Show that the number of pairs $(r_1, r_2) \in S_1 \times S_2$, with $p(r_1, r_2) = 0$, is at most $\min\{(d - k_1)|S_1| + k_1|S_2|, (d - k_2)|S_2| + k_2|S_1|\}$, where for $i = 1, 2$ k_i is the maximum exponent of x_i in $p(x_1, x_2)$. You can assume that $k_1, k_2 \geqslant 1$.

Exercise 6: Linear Programming (5 + 10 points)

(a) Decide which of the following are linear programs and which are not. Do not forget to give a justification.

$$\begin{array}{lrcl} \text{minimize} & 2x+4y & & \\ \text{subject to} & x-3y & = & 5z-3 \\ & x & \geqslant & 0 \\ & y & \leqslant & 0, \end{array} \tag{1}$$

$$\begin{array}{lrcl} \text{maximize} & 8x+2y & & \\ \text{subject to} & x-4y & \leqslant & 8 \\ & x,y & \in & \{0,1\}, \end{array} \tag{2}$$

$$\begin{array}{lrcl} \text{minimize} & 2x+4y-9z & & \\ \text{subject to} & 3x+y & > & 29 \\ & x,y,z & \geqslant & 0. \end{array} \tag{3}$$

(b) Consider a polyhedron of the form $C := \{x \in \mathbb{R}^n | Ax \leqslant 0\}$, $A \in \mathbb{R}^{m \times n}$. Prove that the all-zero vector is the only possible basic feasible solution. Furthermore prove that if C has a basic feasible solution then there exists a vector $c \in \mathbb{R}^n$ such that $c^T x < 0$ for all $x \in C, x \neq 0$. Find such a vector depending on A.