ETH
Eidgenössische Technische Hochschule Zürich
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Algorithms, Probability, and Computing **Midterm Exam** **HS22**

Candidate

First name: ....

Last name: ....

Student ID (Legi) Nr.: ....

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions

1. Check your exam documents for completeness (8 two-sided pages with 5 exercises).
2. You have **2 hours** to solve the exercises.
3. You can solve the exercises in any order. You should not be worried if you cannot solve all exercises! Not all points are required to get the best grade.
4. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints will not be accepted.
5. Pencils are not allowed. Pencil-written solutions will not be reviewed.
6. No auxiliary material allowed. All electronic devices must be turned off and are not allowed to be on your desk. We will write the current time on the blackboard every 15 minutes.
7. Attempts to cheat lead to immediate exclusion from the exam and can have judicial consequences.
8. Provide only one solution to each exercise. Please clearly mark/scratch the solutions that should not be graded.
9. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. Unless stated otherwise, no points will be awarded for unfounded or incomprehensible solutions.**
10. You may use anything that has been introduced and proved in the lecture or in the exercise sessions. You do not need to re-prove it, but you need to state it clearly and concretely. However, if you need something *different* than what we have shown, you must write a new proof or at least list all necessary changes.
11. Write your student-ID (**Legi-number**) on **all** sheets (and your name only on this cover sheet).

	achieved points (maximum)	reviewer's signature
1	(12)	
2	(12)	
3	(12)	
4	(12)	
5	(12)	
Σ	(60)	

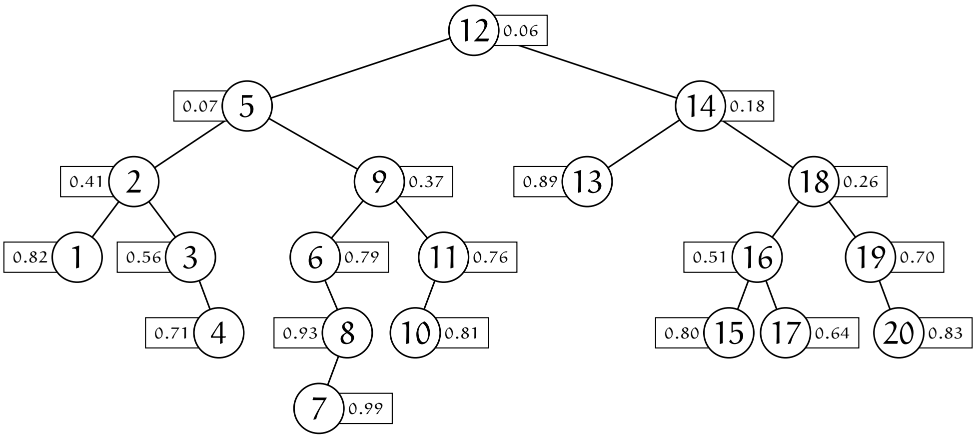
Exercise 1: Five Short Questions (12 points)

No justification is required.
(There are no negative points for wrong answers.)

(a) **(2 points)** In a random search tree on n nodes, what is the probability that all nodes have degree at most 2?

Answer: ....

(b) **(2 points)** Consider the treap shown below. The node 5 has been inserted last. How many rotations were performed during the insertion of node 5?



Answer: ....

(c) **(3 points)** Let S be a set of n non-crossing segments in general position in the plane. Recall the history graph created by the trapezoidal decomposition when inserting the segments in S one by one according to some ordering.

Asymptotically, what is the maximum possible size of the history graph **in the worst-case**?

Answer: ....

Asymptotically, what is the size of the history graph **in expectation**, when the insertion ordering is picked uniformly at random?

Answer: ....

(d) **(3 points)** Consider an LP of the form

$$\text{minimize } \mathbf{c}^\mathsf{T}\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{0},$$

for $\mathbf{c} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}$.

What are the possible basic feasible solution(s)? (1 point)

Answer: ....

Are the following statements true or false? (2 points)

There exists a choice of A and $\mathbf{c}$, such that this LP

(i) is infeasible? [ ] False [ ] True

(ii) has an optimal solution? [ ] False [ ] True

(iii) is unbounded? [ ] False [ ] True

(e) **(2 points)** What is required such that the Ellipsoid method can be applied efficiently to a linear program with an exponential number of constraints?

Answer: ....

....

Exercise 2: A Bad MinCut Algorithm (12 points)

Let G be a multigraph. We consider algorithm $\mathcal{A}$, a modified version of the algorithm BasicMinCut. In each round, instead of contracting a random edge, $\mathcal{A}$ chooses two vertices uniformly at random and merges them (see Figure 1), until the remaining graph has only two nodes. $\mathcal{A}$ then returns the size of the only cut in this graph.

Prove that for any positive integer n, there exists a $2n$-node multigraph G_{2n} for which this algorithm returns a correct minimum cut with an exponentially small probability, i.e.,

$$P[\mathcal{A}(G_{2n}) = \mu(G_{2n})] \in O(\frac{1}{c^n})$$

for some constant $c > 1$.

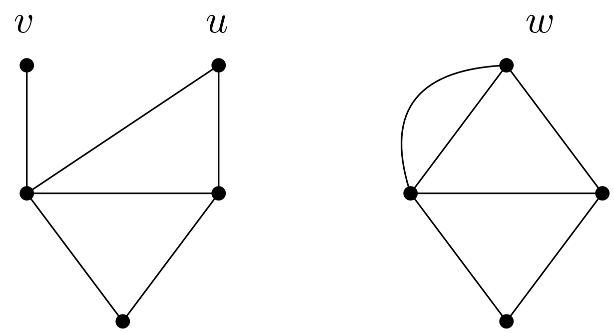


Figure 1: Merging vertices u and v in the left multigraph yield the right multigraph. Note that merging two vertices is equivalent to adding an edge between them and contracting that edge.

Exercise 3: Random Binary Search Tree (12 points)

Let $n \geq 1$. Consider a random binary search tree on n nodes. Compute the expected number of nodes that have both a left and a right child. For full points, give a closed form solution and simplify as much as possible.

Hint: You may use that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ for any $k > 0$.

Exercise 4: Point Location (12 points)

The *bounding box* $b(Q) \in (\mathbb{R} \cup \{-\infty, \infty\})^4$ of a non-empty point set $Q \subset \mathbb{R}^2$ is defined as the tuple of four numbers $(x_{\max}, x_{\min}, y_{\max}, y_{\min})$, where

$$x_{\max} := \max_{p \in Q} p_1,\ x_{\min} := \min_{p \in Q} p_1, y_{\max} := \max_{p \in Q} p_2,\ y_{\min} := \min_{p \in Q} p_2.$$

For $Q = \emptyset$, $b(Q) := (\infty, -\infty, \infty, -\infty)$.

We are now given a point set $P \subset \mathbb{R}^2$ of n points, and we assume that no two points in P share the same x-coordinate, and that no two points in P share the same y-coordinate.

We add the points of P to the empty set one by one in a uniformly random order, yielding a sequence of sets $P_0, P_1, \dots, P_n$, where $P_0 = \emptyset$ and $P_n = P$.

Compute exactly the expected number of bounding box changes during this process. In other words, compute $\mathbf{E}[|\{i : b(P_{i-1}) \neq b(P_i), 1 \leq i \leq n\}|]$. Note that the insertion of the first point is always counted as a bounding box change. You do not need to simplify your solution.

Exercise 5: Linear Programming (12 points)

(a) **(6 points)** Let $\mathbf{c} \in \mathbb{R}^n$ be some fixed vector. Consider the following linear program:

$$\text{minimize } \mathbf{c}^\mathsf{T}\mathbf{x} \text{ subject to } \mathbf{1}^\mathsf{T}\mathbf{x} \geq 3 \text{ and } \mathbf{0} \leq \mathbf{x} \leq \mathbf{2},$$

where $\mathbf{0}, \mathbf{1}$, and $\mathbf{2}$ denote the all-zero, all-ones, and all-twos vectors, respectively. Explicitly[1] describe the set of vectors which are basic feasible solutions of this linear program, and describe an optimal basic feasible solution in terms of $\mathbf{c}$.

(b) **(6 points)** Consider the following linear program in *standard form*,

$$\text{maximize } \mathbf{c}^\mathsf{T}\mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}.$$

Its dual linear program is

$$\text{minimize } \mathbf{b}^\mathsf{T}\mathbf{y} \text{ subject to } A^\mathsf{T}\mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq \mathbf{0}.$$

Let $\mathbf{x}^* \in \mathbb{R}^n$ be a feasible solution of the primal LP and $\mathbf{y}^* \in \mathbb{R}^m$ be a feasible solution of the dual LP. Assuming that there exists an $i \in \{1, \ldots, m\}$ such that ${\mathbf{y}^*}_i > 0$ and $(A\mathbf{x}^*)_i < \mathbf{b}_i$, prove that either x^* or y^* (or both) are not optimal solutions for their respective LPs.
Hint: You may use strong duality.

[1] By "explicitly", we mean that you should find a description of the vectors in this set in terms of their entries, and you should not use the definition of basic feasible solution in your description.