Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
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Algorithms, Probability, and Computing **Exercises KW46** **HS24**

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A: Wed 14–16 CAB G 56

  Group B: Wed 14–16 CAB G 57

  Group C: Wed 16–18 CAB G 56

  Group D: Wed 16–18 CAB G 57

- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

The following exercises will be discussed in the exercise classes on November 13, 2024. Please hand in your solutions via Moodle, no later than 2 pm at November 12.

Exercise 1

Suppose that we have an oracle that, given a system of linear inequalities, decides its feasibility (outputs YES or NO). Design an algorithm that computes a solution of a given system of linear equations and inequalities, provided that one exists, in polynomial time and with polynomially many calls of the oracle.

(a) How can we proceed if there are only equations in the system?

(b) If there is at least one inequality, use the oracle to check if there is a solution satisfying that inequality with equality, and take appropriate actions depending on the outcome.

Exercise 2

Recall the definition of the Subtour LP:

$$\text{minimize } c^\mathsf{T}x \text{ subject to} \qquad \begin{aligned} \sum_{e\in\delta(v)} x_e &= 2 \text{ for all } v \in V \\ \sum_{e\in\delta(S)} x_e &\geq 2 \text{ for all } S \subseteq V \text{ with } \emptyset \neq S \neq V \\ 1 \geq x_e &\geq 0 \text{ for all } e \in E. \end{aligned}$$

(i) Show that a graph $G = (V, E)$ is connected if and only if $\delta(S) \neq \emptyset$ for all $S \subseteq V$, $\emptyset \neq S \neq V$. Prove this using the most basic definition of connectedness: A graph is connected if for any two vertices $v, w \in V$ there is a path from v to w.

(ii) Give a graph G where the subtour LP is feasible but there is no feasible integer solution.

(iii) Assume $|V| \geq 3$. Show that the constraints "$1 \geq x_e$" are redundant in the Subtour LP, i.e. every point $x \in \mathbf{R}^E$ that is feasible w.r.t. all other constraints also satisfies $1 \geq x_e$ for all $e \in E$.

Exercise 3

Let $m \in \mathbf{N}$, $c \in \mathbf{R}^m$, $S \subseteq \mathbf{R}^m$ finite and $P := \operatorname{conv}(S)$. Show that we have

$$\min_{x\in P} c^\mathsf{T}x = \min_{x\in S} c^\mathsf{T}x.$$