

## Solution 1

Let  $x$  be a feasible point of the Tight Spanning Tree LP. Let  $S \subseteq V$ ,  $\emptyset \neq S \neq V$ . We want to show that  $x$  also satisfies the constraint

$$\sum_{e \in \delta(S)} x_e \geq 1. \quad (*)$$

We know

$$\begin{aligned} \sum_{e \in E \cap \binom{S}{2}} x_e &\leq |S| - 1, \\ \sum_{e \in E \cap \binom{V \setminus S}{2}} x_e &\leq |V \setminus S| - 1, \end{aligned}$$

which together is at most  $|V| - 2 = n - 2$ . Due to the first constraint,  $\sum_{e \in E} x_e = n - 1$ , the values  $x_e$  of the remaining edges (those not in  $\binom{S}{2}$  or  $\binom{V \setminus S}{2}$ ) must sum up to at least 1. This is exactly the statement (\*) that we wanted to show.

## Solution 2

Consider the edge set  $E' \subseteq E$  whose characteristic vector corresponds to some feasible  $x \in \{0, 1\}^E$ . Recall that the constraints

$$\sum_{e \in E \cap \binom{S}{2}} x_e \leq |S| - 1, \text{ for all } S \subseteq V, \emptyset \neq S \neq V$$

imply that the subgraphs  $(S, E' \cap S)$  for  $\emptyset \neq S \neq V$  are acyclic. But because  $x$  also satisfies the constraint

$$\sum_{e \in E} x_e = n$$

we know that the graph  $G' = (V, E')$  has a cycle. Therefore this cycle has to be a Hamilton cycle and  $G'$  can not contain other edges since a Hamilton cycle has  $n$  edges. Every characteristic vector of a Hamilton cycle satisfies all the constraints so the characteristic vectors are exactly the Hamilton cycles in  $G$ .

### Solution 3

You might notice that there is a very short direct proof of Lemma 1 in (c). Even though (a) and (b) are longer than (c) the main point is not just proving Lemma 1 but that of practicing the manipulation of systems of inequalities and seeing that the statement is simply one of many instantiations of Farkas lemma.

- (a) The constraints can be interpreted as flow conservation constraints so that for every arc  $e = (u, v)$  the variable  $x_e$  describes the amount of flow from  $u$  to  $v$ . Then for  $v \in V \setminus \{s, t\}$  the constraint

$$\sum_{e \in \delta(v)^+} x_e - \sum_{e \in \delta(v)^-} x_e = 0$$

is saying that the amount of flow incoming to  $v$  is the same as the amount of flow going out from  $v$ . For  $s$  the corresponding constraint is saying that there is more outgoing flow than incoming flow, i.e.,  $s$  is a source of flow, and for  $t$  the constraints dictate that  $t$  is a sink because there is more incoming flow than outgoing flow. It is possible to argue using this flow interpretation but we do not do the proof in such a way.

If there is a directed  $s$ - $t$  path  $P$ , we can set  $x_e = 1$  for every arc  $e$  on the path  $P$  and  $x_e = 0$  for every arc not on  $P$ . It is then easy to see that this constitutes a solution to the system by considering separately constraints corresponding to the vertices that are internal vertices on the path  $P$ , the endpoints  $s, t$  of  $P$  and the vertices not on  $P$ .

To show the other direction let  $\hat{x} \in \mathbb{R}^A$  be a solution to the system given in the exercise description and assume that among all solutions  $\hat{x}$  minimizes the total weight  $\mathbf{1}^T \hat{x}$ . Let  $D' = (V, A')$  be a subgraph of  $D$  so that  $A'$  contains exactly those arcs from  $A$  that have positive weight in  $\hat{x}$ . Observe that  $D'$  is acyclic as otherwise we could reduce the weight on all edges of a cycle by some small amount  $w > 0$  which would result in a feasible solution with smaller total weight than  $\hat{x}$ , a contradiction. The weight  $w$  can be taken to be the minimum weight of any arc on the cycle.

By considering the constraints we observe that  $s$  has at least one outgoing arc in  $D'$  and  $t$  has at least one incoming arc in  $D'$ . Any vertex  $v \in V \setminus \{s, t\}$  that is adjacent to an arc in  $D'$  is adjacent to at least one outgoing and at least one incoming arc. From these properties and from the acyclicity of  $D'$  it then follows that if we start a directed walk from  $s$  in  $D'$  we will necessarily reach  $t$  eventually which proves the existence of an  $s$ - $t$  path since  $D'$  is a subgraph of  $D$ .

- (b) Let us write the equality constraints in a matrix form  $\mathbf{B}\mathbf{x} = \mathbf{b}$  where  $\mathbf{B} \in \mathbb{R}^{V \times A}$ , and  $\mathbf{b} \in \mathbb{R}^V$ . Then for every vertex  $v \in V$  and every arc  $e = (u, w) \in A$  we have that

$$\mathbf{B}_{v,e} = \begin{cases} 1 & \text{if } u = v \\ -1 & \text{if } w = v \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \mathbf{b}_v = \begin{cases} 0 & \text{if } v \in V \setminus \{s, t\} \\ 1 & \text{if } v = s \\ -1 & \text{if } v = t. \end{cases}$$

In other words,  $\mathbf{B}$  is a matrix whose rows are indexed by the vertices  $V$  and whose columns are indexed by the arcs  $A$ . In the column corresponding to the arc  $e = (u, w)$  there is a 1 at the row corresponding to  $u$  and a  $-1$  at the row corresponding to  $w$  and remaining entries in the column are 0.

Using Farkas lemma II from the lecture notes we know that exactly one of the two systems  $\{\mathbf{B}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$  and  $\{\mathbf{B}^T\mathbf{y} \geq \mathbf{0}, \mathbf{b}^T\mathbf{y} < 0\}$  has a solution. To prove Lemma 1 we therefore have to show that the second system has a solution if and only if there exists a strong s-t cut. We can write out the second system as

$$\forall e = (u, w) \in A : \quad \mathbf{y}_u - \mathbf{y}_w \geq 0 \text{ and} \quad (1) \\ \mathbf{y}_s - \mathbf{y}_t < 0.$$

Assume first that  $S$  is a strong s-t cut. Then by setting  $\mathbf{y}_v = 0$  for  $v \in S$  and  $\mathbf{y}_v = 1$  for  $v \in V \setminus S$  satisfies the constraints in (1) in particular because the constraints corresponding to the arcs in the cut  $C(S)$  are satisfied due to the strong s-t cut property, the constraints corresponding to edges within  $S$  and  $V \setminus S$  hold with equality and for the last constraint we have  $\mathbf{y}_s - \mathbf{y}_t = 0 - 1 = -1 < 0$  as it should.

To show the other direction assume that there exists a solution  $\hat{\mathbf{y}}$  to (1). Define  $S := \{v \in V \mid \hat{\mathbf{y}}_v \leq \hat{\mathbf{y}}_s\}$  as the set of vertices that are assigned a value at most  $\hat{\mathbf{y}}_s$  in  $\hat{\mathbf{y}}$ . We claim that  $S$  is a strong s-t cut. Firstly, by definition  $s \in S$  and by the second inequality of (1) we have  $t \notin S$  so  $S$  is an s-t cut. Also from the definition of  $S$  we observe that for every  $u \in S$  and every  $w \in V \setminus S$  it holds that  $\hat{\mathbf{y}}_u - \hat{\mathbf{y}}_w < 0$ . Since  $\hat{\mathbf{y}}$  is a solution to (1) this implies that there can be no arcs  $(u, w) \in A$  with  $u \in S, w \in V \setminus S$  and therefore  $S$  is a strong s-t cut. This concludes the proof.

- (c) If there is a directed s-t path  $P$ , then there cannot exist a strong s-t cut since for any set  $S \subseteq V$  with  $s \in S$  and  $t \notin S$  there is at least one arc of  $P$  that goes from  $S$  to  $V \setminus S$ . If there is no directed s-t path, we let  $S$  be the set of all vertices reachable from  $s$  by a directed path. Then  $S$  is a strong s-t cut because no vertex of  $V \setminus S$  is reachable from a vertex of  $S$ .

## Solution 4

Assume that the matrix  $C$  is wrong in exactly the  $i$ -th row compared to the correct product  $AB$ . There is at least one error in this row but perhaps there is more than one error. We define  $D = AB - C$  and write  $D = (d_{ij})$ . This is a zero-matrix except in the  $i$ -th row there are some ones. We may assume that the  $d_{ij} = 1$ . Take a vector  $\mathbf{x} \in_{\text{u.a.r.}} \{0, 1\}^n$  and compute the product  $D\mathbf{x}$ . Every entry but the  $i$ -th one is zero. We look at the  $i$ -th entry of this product and split this up

$$(D\mathbf{x})_i = \sum_{k=1}^n d_{ik}x_k = \underbrace{\sum_{k=1, k \neq j}^n d_{ik}x_k}_{=:S} + d_{ij}x_j,$$

where the  $S$  is some number, either 0 or 1. The probability that we will detect the error is exactly the probability that  $(Dx)_i = 1$ . Let  $x_j$  be the last entry of  $x$  that we choose randomly. Then independent of the value of  $S$  we see that  $(Dx)_i$  is 0 with probability  $\frac{1}{2}$ , and it is 1 with probability  $\frac{1}{2}$ . This means that the probability of detecting an error is exactly  $\frac{1}{2}$ .

REMARK: What we proved here is that indeed for any vectors  $a, b \in \text{GF}(2)^n$ ,  $a \neq b$  fixed and for  $x \in_{\text{u.a.r.}} \text{GF}(2)^n$

$$\Pr[a^T x = b^T x] = \frac{1}{2},$$

where  $^T$  stands for the transpose of a vector.

## Solution 5

Let  $i, j \in \{1..n\}$  be indices such that  $A_{ij}$  is nonzero. Consider the  $i$ -th entry  $(Ax)_i$  of the matrix-vector product. It calculates as

$$(Ax)_i = \sum_{k \in \{1..n\} \setminus \{j\}}^n A_{ik} x_k + A_{ij} x_j.$$

Since the  $x_i$  are being chosen independently of one another, we may prescribe any order in which they are evaluated; let us evaluate  $x_j$  last. Once all terms in the sum are being fixed, then it takes a fixed real value  $s$ . At most one out of the possible choices for  $x_j$  can yield  $A_{ij} x_j = -s$  and so the probability that this happens is at most  $1/3$ , yielding the claim.

## Solution 6

Let  $\{a_1, \dots, a_d\} \subseteq S$  be a set of  $d$  elements. We define the polynomial  $p(x_1, \dots, x_n)$  as

$$p(x_1, \dots, x_n) := (x_1 - a_1)(x_1 - a_2) \cdots (x_1 - a_d).$$

Note that the only variable occurring in this polynomial is  $x_1$ , and the degree of the polynomial is  $d$ .

This polynomial evaluates to zero if and only if  $x_1 \in \{a_1, \dots, a_d\}$ . The other variables  $x_2, \dots, x_n$  can be set to arbitrary values in  $S$ . Therefore, the number of  $n$ -tuples  $(r_1, \dots, r_n) \in S^n$  with  $p(r_1, \dots, r_n) = 0$  is exactly

$$\underbrace{d}_{\text{choices for } r_1} \times \underbrace{|S|}_{\text{choices for } r_2} \times \cdots \times \underbrace{|S|}_{\text{choices for } r_n},$$

which is  $d \cdot |S|^{n-1}$ .