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1/31/2008

# *Algorithms, Probability, and Computing* — *Fall 07 Exam*

**Candidate:**

First name: ................................................................................

Last name: ................................................................................

Registration Number: ................................................................................

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I confirm with my signature that I take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ................................................................................

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**General remarks and instructions:**

1. You can solve the 6 assignments in any order.
2. Check your exam documents for completeness (1 title sheet and 1 sheet containing 6 assignments).
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints will not be accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. No additives (Hilfsmittel) allowed.
6. Attempts to defraud yield to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in articulative sentences and keywords. Unfounded or incomprehensible solutions will not be awarded.**

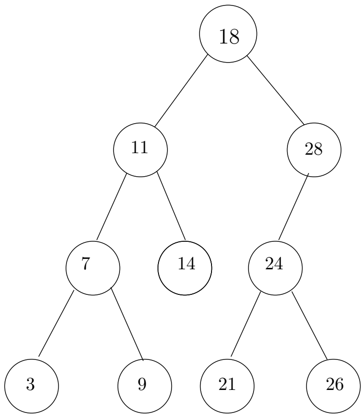
   **You can write your solution in English or German.**
9. Make sure to write your name on all the sheets.

Good luck!

| | achieved points (maximum) | reviewer's signature |
|---|---|---|
| 1 | (12) | |
| 2 | (12) | |
| 3 | (12) | |
| 4 | (12) | |
| 5 | (12) | |
| 6 | (12) | |
| $\Sigma$ | (72) | |

## Assignment 1

1. We consider a random search tree on $n$ nodes. Recall that by definition a node $u$ is a *descendant* of node $v$ if $u$ is in the subtree rooted at $v$ (possibly $u = v$). For example, in the tree below the node with key 11 has 5 descendants.




   (a) Give a recurrence for the expected number of nodes with at least 4 descendants.

   (b) Compute the probability that the element of rank $i \in \{1, ..., n\}$ has a left child.

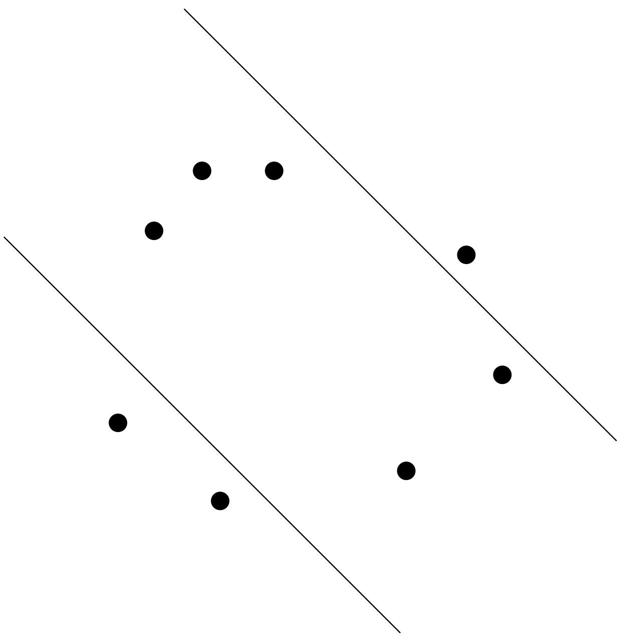
2. Define

$$x_n := 7 \cdot n + \sum_{i=1}^{n-1} x_i + x_{n-i},$$

   for $n \geq 2$ and $x_1 := 7$. Solve this recurrence. You can use the fact that $\sum_{i=0}^{n} 3^i = \frac{3^{n+1}-1}{2}$.

## Assignment 2

Let $l_1$ and $l_2$ be two parallel and different lines in the plane. We call the area between $l_1$ and $l_2$ a *strip*. Let $P \subseteq \mathbb{R}^2$ be a set of $n$ points in convex position. Describe a data structure to store the $n$ points in $P$ which uses $O(n)$ space and allows to query the number of points in a strip in time $O(\log(n))$. The strip is given by the two surrounding lines. Describe the data structure and the query algorithm *in detail*. Show how your algorithm works on the example below.



## Assignment 3

For an integer $k \geq 1$ we define the network $N_k := (G_k, s, t, c)$ as follows. The directed graph $G_k = (V, E)$ has vertex set $V := \{s, t, u_1, \dots, u_{4k}, v_1, \dots, v_{4k}\}$ and edge set $E := A \cup B \cup C \cup D$ where

$$A := \{(s, u_i) : i = 1, \dots, 4k\},$$

$$B := \{(u_i, v_j) : i \in \{1, \dots, 4k\}, j \in \{1, \dots, k\}\},$$
$$C := \{(u_i, v_j) : i \in \{3k+1, \dots, 4k\}, j \in \{1, \dots, 4k\}\},$$

and

$$D := \{(v_i, t) : i = 1, \dots, 4k\}.$$

We define the edge capacities as $c(e) := 1$ for all $e \in E$.

1. Draw $N_1$ and find a maximum flow in it.
2. Prove that the value of a maximum flow in $N_k$ equals $2k$.

## Assignment 4

Describe Karger's Algorithm for the minimum cut problem. In doing so answer the following questions: What is the basic operation? How are this basic operation and the minimum cut number connected? Bound the probability that after the basic operation the minimum cut number will not change? Explain how this idea can be used to obtain a (randomized) algorithm for the minimum cut problem. What success probability does a straightforward implementation of this idea have. How can you improve it?

## Assignment 5

For an integer $n \geq 1$ let $p_1(x_1, ..., x_n), ..., p_n(x_1, ..., x_n)$ be $n$ polynomials over GF(5).[1] Every polynomial has total degree at most 4. We want to decide whether there exists $i \in \{1, ..., n\}$ such that $p_i$ is not identical 0. We describe a round of our algorithm next: Uniformly at random choose $i \in \{1, ..., n\}$ and uniformly at random choose $a \in \mathrm{GF}(5)^n$. If $p_i(a) \neq 0$ then stop and accept. Our algorithm $A(t)$ repeats this step $t$ times. If the algorithm did not stop and accept in one of the rounds, it rejects.

Assume that there exists $i \in \{1, ..., n\}$ such that $p_i$ is not identical 0.

1. Show that the probability that $A(1)$ accepts is at least $\frac{1}{5n}$.
2. Show that the probability that $A(5n)$ accepts is at least $\frac{1}{2}$. You can assume that the claim in (1) holds.

## Assignment 6

1. Let $a, b \in \mathrm{GF}(5)^n$.[1] Assume $a \neq b$. Show that

$$\Pr_{x \in \mathrm{GF}(5)^n}(a^T x \neq b^T x) = 4/5.$$

2. Let $A$ and $B$ be two different $n \times n$ matrices over GF(5). Show that

$$\Pr_{x,y \in \mathrm{GF}(5)^n}((Ax)^T y \neq (Bx)^T y) \geq 16/25.$$

   You can assume that the claim in (1) holds.

[1] Recall that GF(5) is the finite field of order 5, e.g. the numbers $\{0, ..., 4\}$ with addition and multiplication modulo 5.