Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
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Algorithms, Probability, and Computing — Final Exam — HS14

Candidate

First name: ....

Last name: ....

Student ID (Legi) Nr.: ....

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions

1. Check your exam documents for completeness (2 cover pages and 3 pages with 6 exercises).
2. You can solve the six exercises in any order. You should not be worried if you cannot solve all exercises! Not all points are required to get the best grade.
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints will not be accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. No auxiliary material allowed. All electronic devices must be turned off and are not allowed to be on your desk.
6. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. Unless stated otherwise, no points will be awarded for unfounded or incomprehensible solutions. You can write your solutions in English or German.**
9. You may use anything that has been introduced and proved in the lecture or in the exercise sessions without reproving it (unless you are explicitely asked to reproduce parts of a certain proof). However, if you need something *different* than what we have shown, you must write a new proof or at least list all necessary changes.
10. Write your student-ID (**Legi-number**) on **all** sheets (and your name only on this cover sheet).

Good luck!

	achieved points (maximum)	reviewer's signature
1	(20)	
2	(20)	
3	(20)	
4	(20)	
5	(20)	
6	(20)	
Σ	(120)	

Exercise 1: Solving Recurrences (20 points)

Find closed forms for the three recurrences defined below.

(a) $$a_n := \begin{cases} 2, & \text{for } n = 1, \\ 3 + 4\sum_{i=1}^{n-1} a_i, & \text{for } n \geqslant 2. \end{cases}$$

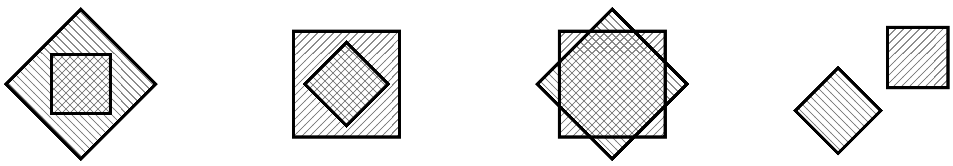
(b) $$b_n := \begin{cases} 2, & \text{for } n = 1, \\ 3 + 4\sum_{i=1}^{n-1} (-1)^i b_i, & \text{for } n \geqslant 2. \end{cases}$$

(c) $$c_n := \begin{cases} 2, & \text{for } n = 1, \\ 3 + 4\sum_{i=1}^{n-1} (-1)^{n-i} c_i, & \text{for } n \geqslant 2. \end{cases}$$

Exercise 2: Intersecting Convex Polygons (20 points)

Let P and Q be two distinct convex polygons in the plane (for the purpose of this exercise, by *polygon* we mean the set of all boundary and interior points). They are both given as sorted arrays that contain the respective vertices in clockwise order. Design a deterministic algorithm that runs in time $O(n)$ (where n is the overall number of vertices) and which decides which one of the following four cases holds: (i) P contains Q, (ii) Q contains P, (iii) P and Q intersect but neither polygon contains the other, (iv) P and Q are disjoint (see also Figure 1).

REMARK: You can make any general position assumptions in order to avoid corner cases, but you should always make such assumptions explicit.



(i) (ii) (iii) (iv)

Figure 1: The four cases to be distinguished in exercise 2 with $P =$  and $Q =$ .

Exercise 3: Schwartz-Zippel and Perfect Matchings (20 points)

(a) Let $f(x, y, z) := 2x^2y - 3zx + 8y^2 - 2yx^2 + 4yz - z \pmod{11}$ and let x, y, z be chosen independently and uniformly at random from the set $\{1, 2, \dots, 10\}$. What is the smallest upper bound on $\Pr[f(x, y, z) = 0]$ that you can get by using the Schwartz-Zippel Theorem?

For the remaining tasks, let $G = (U \cup V, E)$ be a bipartite graph with partite sets $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$. Moreover, recall the $n \times n$ matrix $A = (a_{i,j})_{i,j=1}^n$, which ist defined as follows (recall also that the x_{ij} are variables).

$$a_{ij} := \begin{cases} x_{ij}, & \text{if } \{u_i, v_j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Prove that $\det(A)$ is not the zero polynomial (in the variables x_{ij}) if and only if there exists a perfect matching in G.

(c) Let us define a new matrix B which is the same as A, but we substitute 1 for each variable x_{ij}. Find a concrete bipartite graph G which does have a perfect matching but for which $\det(B) = 0$ holds.

(d) Describe the algorithm which we used to test whether a bipartite graph contains a perfect matching in time $O(n^3)$. Analyse its running time, its success probability, and explain why it does not always give a correct answer.

Remark: You do not have to explain how repeated invocations of this algorithm can further reduce the error probability.

Exercise 4: Ellipsoid Method (20 points)

Recall from the lecture that the ellipsoid method cannot be used to solve arbitrary linear programs directly. Instead, it only solves a so called *relaxed feasibility problem*, which has four inputs A, b, R and ϵ.

(a) Give a precise definition of the relaxed feasibility problem in n variables. That is, explain the meaning of all four inputs and specify all correct outputs.

(b) Give a rough outline of the main steps that the ellipsoid method takes when solving a relaxed feasibility problem and argue why it returns a correct solution.

(c) Without giving a proof, for any fixed n, what is the crucial relation between two ellipsoids that correspond to two consecutive iterations of the ellipsoid method (crucial in the sense that it allows us to prove an upper bound on the number of iterations)?

Remark: It is not important to remember all involved numbers exactly, as long as they make sense qualitatively and as long as they allow you to solve task (d).

(d) Prove an upper bound of $O(\log(R/\epsilon))$ on the number of iterations until the ellipsoid method terminates by using what you wrote down in task (c).

Remark: Usually, this upper bound also depends on n. But to make the task simpler we again assume that n is a fixed constant.

Exercise 5: Relaxation and Approximation (20 points)

Let $G = (V, E)$ be the complete graph on n vertices and let us assign a non-negative cost c_e to every edge $e \in E$. We recall the Loose Spanning Tree LP of G from the lecture (recall also that $\delta(S)$ is the set of edges in G that go from S to $V \setminus S$):

$$\begin{aligned} \min \quad & \textstyle\sum_{e\in E} c_e x_e \\ \text{subject to} \quad & \textstyle\sum_{e\in E} x_e = n-1 \\ & \textstyle\sum_{e\in\delta(S)} x_e \geqslant 1 \quad \text{for all } S \subseteq V,\ \emptyset \neq S \neq V \\ & 0 \leqslant x_e \leqslant 1 \quad \text{for all } e \in E \end{aligned}$$

A *spanning path* is a subset $P \subseteq E$ so that there exists a permutation $v_1, v_2, \ldots, v_n$ of V with $P = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$.

(a) Add additional constraints to the Loose Spanning Tree LP of G so that its integral solutions are the characteristic vectors of all spanning paths.

(b) Suppose you want to find the optimal solution of your LP from task (a) in time polynomial in n. Is that possible by using any of the algorithms we have seen in the lecture?

(c) Prove or disprove: The integrality ratio of your LP from task (a) is always equal to 1.

Exercise 6: Fourier Transform (20 points)

First, you might want to recall the following from the lecture, which holds for any function $f\colon \{-1,1\}^n \to \{-1,1\}$.

Definition: The Fourier coefficient $\hat{f}(S)$ is defined for every set $S \subseteq \{1,\dots,n\}$ as

$$\hat{f}(S) := \frac{1}{2^n} \sum_{x \in \{-1,1\}^n} f(x) \cdot \chi_S(x).$$

Lemma: For any function f we have

$$\sum_{S \subseteq \{1,\dots,n\}} \hat{f}^2(S) = 1.$$

Let f be arbitrary. Suppose further that z is an input chosen uniformly at random from the set $\{-1,1\}^n$. In this exercise we are interested in the probability $\Pr[f(z) = f(-z)]$, where by $-z$ we mean the vector obtained by negating all components of the vector z.

(a) Prove that

$$\Pr[f(z) = f(-z)] = \frac{1}{2^n} \sum_{x \in \{-1,1\}^n} \frac{1 + f(x) \cdot f(-x)}{2}.$$

(b) Using what you have already shown in (a), prove that

$$\Pr[f(z) = f(-z)] = \frac{1}{2} \left(1 + \sum_{S \subseteq \{1,\dots,n\}} \hat{f}^2(S) \cdot (-1)^{|S|} \right).$$

For the remaining tasks, assume that f has the following special property: For every set $S \subseteq \{1,\dots,n\}$ of odd size, we have

$$\Pr[f(z) = \chi_S(z)] = \frac{1}{2}.$$

(c) For every set $S \subseteq \{1,\dots,n\}$ of odd size, prove that $\hat{f}(S) = 0$.

(d) Finally, prove that $\Pr[f(z) = f(-z)] = 1$.