Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Thomas Holenstein, Angelika Steger, Emo Welzl

Algorithms, Probability, and Computing — Final Exam — HS15

Candidate

First name: ....

Last name: ....

Student ID (Legi) Nr.: ....

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions

1. Check your exam documents for completeness (2 cover pages and 5 pages with 6 exercises).
2. You can solve the six exercises in any order. You should not be worried if you cannot solve all exercises! Not all points are required to get the best grade.
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints will not be accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. No auxiliary material allowed. All electronic devices must be turned off and are not allowed to be on your desk.
6. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. Unless stated otherwise, no points will be awarded for unfounded or incomprehensible solutions. You can write your solutions in English or German.**
9. You may use anything that has been introduced and proved in the lecture or in the exercise sessions without reproving it (unless you are explicitly asked to reproduce parts of a certain proof). However, if you need something *different* than what we have shown, you must write a new proof or at least list all necessary changes.
10. Write your student-ID (**Legi-number**) on **all** sheets (and your name only on this cover sheet).

Good luck!

	achieved points (maximum)	reviewer's signature
1	(20)	
2	(20)	
3	(20)	
4	(20)	
5	(20)	
6	(20)	
Σ	(120)	

Exercise 1: Assorted Tasks (20 points)

(a) Solve the following recurrence for all $n \in \mathbb{N}$.

$$a_n = \begin{cases} 3, & \text{for } n = 0 \\ \frac{1}{n} \sum_{i=0}^{n-1} 2a_i, & \text{for } n \geqslant 1. \end{cases}$$

(b) Let $T : \mathbb{N} \to \mathbb{R}_{>0}$ be a function such that for all $n \in \mathbb{N}$: $T(n) \leqslant 5n^2 + 2T\left(\lceil \frac{n}{2^{\frac{1}{3}}} \rceil\right)$.

Prove that $T(n) = O(n^3 \log n)$, i.e., there exists some constant $k > 0$, such that $T(n) \leqslant kn^3 \log n$ for all $n \in \mathbb{N}$.

HINT: $1.2 < 2^{\frac{1}{3}} < 1.3$.

(c) Prove or disprove: There exists a graph $G = (V, E)$, with $E \neq \emptyset$, that has an odd number of Pfaffian orientations.

(d) Prove or disprove: Let S_n be the set of all permutations on $[n] = \{1, 2, \dots, n\}$. Let $\pi \in S_n$ and $C_1, \dots, C_k$ the set of odd cycles in π. Given only $C_1, \dots, C_k$ and n, one can determine $\text{sign}(\pi)$.

Exercise 2: Lovely Nodes (20 points)

For a binary tree on the nodes $[n] = \{1, 2, \dots, n\}$ we say a node is *lovely* if its left or its right subtree (or both) have exactly one node. Let X_n be the number of lovely nodes in a random search tree and $x_n = \mathbb{E}[X_n]$.

(a) Compute x_0, x_1, x_2 and x_3.

(b) Find a recurrence formula for x_n, for all $n \geqslant 4$.

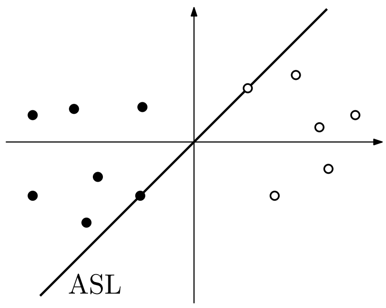
(c) Solve the above recurrence formula for all $n \geqslant 4$.

HINT: Observe that the formula is different for $n = 3$.

Exercise 3: Once More, Separating Lines (20 points)

We are given a set of $n > 0$ black points and $m > 0$ white points in the plane, called B and W. For simplicity we assume general position, i.e., no three points of $B \cup W$ lie on a line. Furthermore assume that all black points have negative x-coordinate and all white points have positive x-coordinate. We call a line ℓ *almost separating* (ASL), if

1. One black point and one white point lie on ℓ, and
2. all other black points are above and all other white points are below ℓ.



Throughout you can use without proof that the ASL exists and is unique.

HINT: You can use parts (a) and (b) of the exercise in the later parts, even if you are not able to solve them.

(a) Consider the special case where we only have one white point, i.e., $W = \{w_1\}$. Show that the ASL of B and w_1 can be computed in time $O(n)$.

The goal now it to compute the ASL of B and W for general $m > 0$. Let $W = \{w_1, w_2, \ldots, w_m\}$ be a u.a.r. permutation of W.
Let $W_i := \{w_1, \ldots, w_i\}$ and ℓ_i be the ASL of B and W_i. Consider the following algorithm: Compute ℓ_1 in time $O(n)$ as in part (a). In step $i \geqslant 2$ we are given ℓ_{i-1}. Check whether ℓ_{i-1} is still the ASL for B and W_i, if not, find the new ASL ℓ_i.

(b) For $i \geqslant 2$ prove that if w_i is below ℓ_{i-1}, then $\ell_i = \ell_{i-1}$. Furthermore prove that if w_i is above ℓ_{i-1}, then w_i lies on ℓ_i.

Prove that therefore one can check in constant time whether $\ell_{i-1} = \ell_i$ and if $\ell_{i-1} \neq \ell_i$, then ℓ_i can be found in time $O(n)$.

(c) Prove that the worst case running time of the algorithm is $\Omega(nm)$, that is give an example, where the algorithm has to update ℓ_i in every step, i.e., $\ell_i \neq \ell_{i-1}$ for all $i \geqslant 2$. For simplicity you can drop the general position assumption to construct the example.

(d) Show that the expected running time of the algorithm is $O(m + n \log m)$.

Exercise 4: Duality (20 points)

Consider the following LP.

$$\begin{array}{lrcl} \text{maximize} & c^T x & & \\ \text{subject to} & Ax & \leqslant & b \\ & x & \geqslant & 0, \end{array} \qquad (1)$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.
From the lecture we know that its dual is defined as

$$\begin{array}{lrcl} \text{minimize} & b^T y & & \\ \text{subject to} & A^T y & \geqslant & c \\ & y & \geqslant & 0. \end{array} \qquad (2)$$

(a) Show that the dual of

$$\begin{array}{lrcl} \text{maximize} & c^{\mathsf{T}}x & & \\ \text{subject to} & Ax & \leqslant & b, \end{array} \tag{3}$$

is

$$\begin{array}{lrcl} \text{minimize} & b^{\mathsf{T}}y & & \\ \text{subject to} & A^{\mathsf{T}}y & = & c \\ & y & \geqslant & 0, \end{array} \tag{4}$$

by changing (3) to the form (1) and using the definition. Make sure that your calculation steps are clear!

(b) Every LP can have an optimal solution (OS), be unbounded (UB) or infeasible (IF). In the following mark every possible pairing with (Y) and every impossible pairing with (N), e.g., mark the box corresponding to (OS, OS) with (Y), if it is possible that both the primal and the dual have an optimal solution or mark it with (N) if it is not possible. You do not need to give a proof of correctness.

		(D) OS	UB	IF
	OS			
(P)	UB			
	IF			

(c) You can assume that the following LP has optimal value 3.

$$\begin{array}{lrcl} \text{maximize} & x_2 & & \\ \text{subject to} & -x_1 + 2x_2 & \leqslant & 4 \\ & x_1 & \leqslant & 3 \\ & x_1 + x_2 & \leqslant & 5 \\ & x_1, x_2 & \geqslant & 0, \end{array} \tag{5}$$

What is the optimal value of the following LP? Argue why your solution holds.

$$\begin{array}{lrcl} \text{maximize} & 4y_1 + 5y_2 + 3y_3 & & \\ \text{subject to} & -y_1 + y_2 + y_3 & \geqslant & 0 \\ & 2y_1 + y_2 & \geqslant & 1 \\ & y_1, y_2, y_3 & \geqslant & 0, \end{array} \tag{6}$$

Exercise 5: Approximation Algorithm (20 points)

Let $G = (V, E)$ be a graph with $|V| = n$. A matching of G is a set $E' \subseteq E$ such that every vertex $v \in V$ is incident to at most one edge in E'. A maximum matching is a matching of maximum size.

(a) Write down the constraints from above as an integer program (IP) that finds the maximum matching, in terms of the variables x_e, $e \in E$. (An integer program is a linear program with the additional constraints $x_e \in \mathbb{N}$.)

Give the corresponding LP relaxation (LP) of the maximum matching problem.

(b) For every $n \geqslant 3$, give an example of a graph, where (LP) has a larger solution than (IP).

(c) Add constraints to (LP) to obtain an LP (LP′) that has the following properties.

- All solutions that correspond to matchings are still solutions of (LP′).
- The solutions of (LP) that you found in part (b) are not feasible for (LP′).

You do not need to show that (LP′) has no solution larger than (IP).

Exercise 6: Linear and Affine Functions (20 points)

Let GF(3) be the field of integers modulo 3. A function $f : GF(3)^n \to GF(3)$ is *linear* if $f(x) + f(y) = f(x + y)$ for all $x, y \in GF(3)^n$, where $x + y$ is the componentwise addition in $GF(3)^n$ and $f(x) + f(y)$ is the addition in GF(3).
A function $g : GF(3)^n \to GF(3)$ is called *affine* if there exists $s \in GF(3)^n$ and $b \in GF(3)$ such that

$$g(x) = \sum_{i=1}^{n} s_i x_i + b.$$

From now on let $h : GF(3)^n \to GF(3)$ be some function, $g : GF(3)^n \to GF(3)$ an affine function. Furthermore let $x^* \in GF(3)^n$ fixed and $x, y, z \in_{u.a.r.} GF(3)^n$.

(a) Let g and h be such that $\Pr[h(x) = g(x)] \geqslant 1 - \delta$. Prove that

$$\Pr[g(x^*) = h(z) + h(x^* + y) - h(y + z)] \geqslant 1 - 3\delta.$$

HINT: You can use without proof that f is linear if and only if there exists $s \in GF(3)^n$ such that $f(x) = \sum_{i=1}^{n} s_i x_i$.

(b) Let $g(x) = \sum_{i=1}^{n} s_i x_i + b$. Prove that for $(s, b) \neq (0, 0)$

$$\Pr[\sum_{i=1}^{n} s_i x_i + b \neq 0] \geqslant \frac{2}{3}.$$

(c) Give an example where the bound of (b) is tight, i.e., find a pair $(s, b) \neq (0, 0)$ such that

$$\Pr[\sum_{i=1}^{n} s_i x_i + b \neq 0] = \frac{2}{3}.$$

Furthermore give an example where the bound is strict, i.e., find a pair $(s, b) \neq (0, 0)$ such that

$$\Pr[\sum_{i=1}^{n} s_i x_i + b \neq 0] > \frac{2}{3}.$$

Do not forget to give a justification.