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11/08/2007

Algorithms, Probability, and Computing *Fall 07*
Mid-Term Exam

Candidate:

First name: ....

Last name: ....

Registration Number: ....

I attest with my signature that I could have taken the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions:

1. You can solve the 6 assignments in any order. **You should not be worried if you cannot solve all the assignments** (best grade with at least 54 points, i.e. $4\frac{1}{2}$ assignments).
2. Check your exam documents for completeness (1 cover sheet and 3 sheets containing 6 assignments).
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints are not accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. No additives allowed.
6. Attempts to defraud yield to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in articulative sentences and keywords. Unfounded or incomprehensible solutions will not be awarded. You can write your solution in English or German.**
9. Make sure to write your name on all the sheets.

Good luck!

	achieved points (maximum)	reviewer's signature
1	(12)	
2	(12)	
3	(12)	
4	(12)	
5	(12)	
6	(12)	
Σ	(72)	

Assignment 1

Let S_n be the set of all permutations of $\{1, ..., n\}$ and for $\pi \in S_n$ let I_n be the number of pairs (i, j) such that $1 \leq i < j \leq n$ and $\pi_i > \pi_j$. For example, $I_5 = 4$ for $\pi = 25134$. We uniformly at random select π from S_n.

1. Give a recursive equation for $\mathbb{E}(I_n)$.
2. Determine $\mathbb{E}(I_n)$. (You need not necessarily go through the recurrence in (1) for that.)

Assignment 2

We consider a convex polygon P with vertices a, b, c, d (see the below figure).

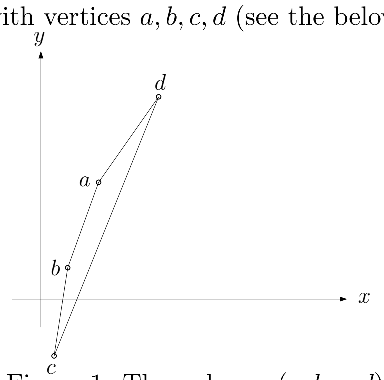
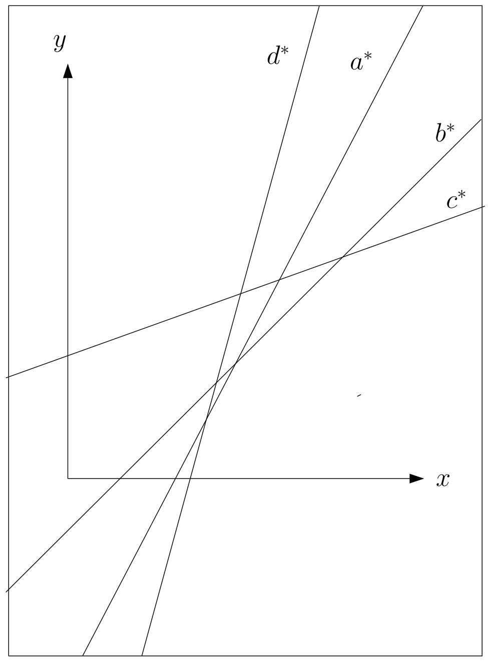
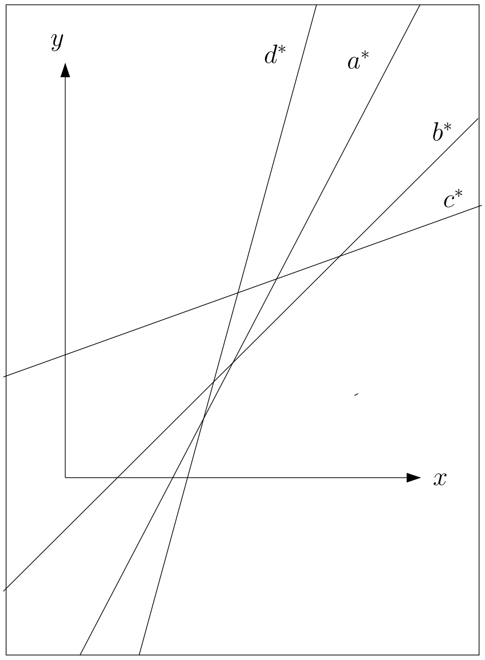


Figure 1: The polygon (a, b, c, d)

1. Let S be the set of duals of the lines which do not intersect the polygon (a, b, c, d). Draw S in the picture below. (The lines a^*, b^*, c^*, d^* are the duals of a, b, c, d.) Give a short explanation.

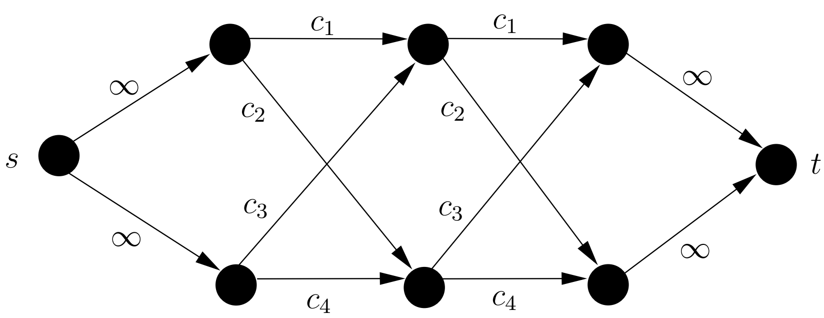


2. Draw the set of tangents of the polygon (a, b, c, d) in the picture below.



Assignment 3

Compute a maximum flow and its value in the following network. The capacities are all non-negative. You will have to distinguish between several cases.



Assignment 4

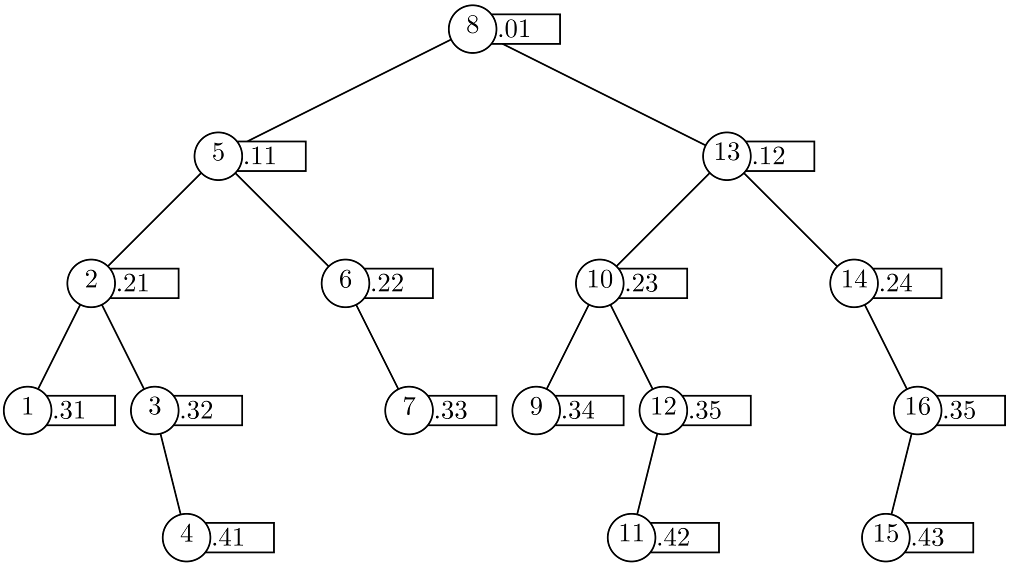
1. We have given a treap which stores the keys $\{1, 7, 8, 9, 15, 21, 22, 23, 30, 31, 33, 35, 40, 51\}$. We search for 32. Which of the sequences can not be a search sequence (sequence of keys 32 is compared to during the search)? Give a short explanation.

   (a) $7, 15, 35, 31, 23, 25$

   (b) $51, 40, 8, 9, 15, 22, 23$

   (c) $8, 9, 31, 51, 40, 30, 33$

2. How many rotations are necessary to remove element 13 in the following treap?



Assignment 5

Let L_n be the number of nodes with empty left subtree in a random search tree of n elements. Give a recurrence for $\mathbb{E}(L_n)$ and solve it.

Assignment 6

We have given a network $G = (V, E, s, t, c)$ with integer capacities selected independently and uniformly at random from $\{0, 1, ..., w\}$, i.e., $\Pr(c(e) = i) = \frac{1}{w+1}$ for all $e \in E$ and $i \in \{0, 1, ..., w\}$. Let M be the value of the maximum flow in G. Assume that for some $U \subseteq V$ with $s \in U$ and $t \notin U$ it holds that

$$|\{(x, y) \in E : x \in U, y \in V \setminus U\}| \leq \Delta. \tag{$*$}$$

1. Prove that $\mathbb{E}(M) \leq \frac{w}{2} \cdot \Delta$.
2. Prove that for every integer $\Delta \geq 1$ there exists a network with $\mathbb{E}(M) = \Delta \cdot \frac{w}{2}$ and for which property $(*)$ holds. Multiple edges are allowed here.