ETH

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Mohsen Ghaffari, Angelika Steger, Emo Welzl, Peter Widmayer

Algorithms, Probability, and Computing **Midterm Exam** **HS16**

Candidate

First name: ....

Last name: ....

Student ID (Legi) Nr.: ....

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ....

General remarks and instructions

1. Check your exam documents for completeness (pages numbered from 1 to 11, with one exercise on every page except for the two cover pages).
2. You can solve the exercises in any order. You should not be worried if you cannot solve all exercises! Not all points are required to get the best grade.
3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints will not be accepted.
4. Pencils are not allowed. Pencil-written solutions will not be reviewed.
5. No auxiliary material allowed. All electronic devices must be turned off and are not allowed to be on your desk. We will write the current time on the blackboard every 15 minutes.
6. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.
7. Provide only one solution to each exercise. Cancel invalid solutions clearly.
8. **All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. Unless stated otherwise, no points will be awarded for unfounded or incomprehensible solutions. You can write your solutions in English or German.**
9. You may use anything that has been introduced and proved in the lecture or in the exercise sessions without reproving it. However, if you need something *different* than what we have shown, you must write a new proof or at least list all necessary changes.
10. Write your student-ID (**Legi-number**) on **all** sheets (and your name only on this cover sheet).

Good luck!

	achieved points (maximum)	reviewer's signature
1	(20)	
2	(20)	
3	(20)	
4	(15)	
Σ	(75)	

Exercise 1: Multiple Choice (20 points)

Consider the following four claims/questions and mark the corresponding boxes. Grading: 2 points for a correct marking without a correct justification, 5 points for a correct marking with a correct short justification, and -2 points for a wrongly marked box. You will receive a non-negative total number of points for the whole exercise in any case.

(a) Suppose that a sequence $(a_n)_{n\geq 1}$ of strictly positive numbers satisfies $a_n^2 = \sum_{i=1}^{n-1} a_i^2$ for $n \geq 2$. Then $a_n = \Theta(\sqrt{2}^n)$.

[ ] False [ ] True

Justification: ....

....

(b) The expected number of comparisons done by randomized quicksort can be $\binom{n}{2}$ in the worst case.

[ ] False [ ] True

Justification: ....

....

(c) Let $p = x_1(x_2 + x_3x_4 - x_5x_6) \in \mathbb{Q}[x_1, \ldots, x_6]$.
If $X \in_{\text{u.a.r.}} \{0,1\}^6$ then $\Pr[p(X) = 0] \leq \frac{1}{2}$.

[ ] False [ ] True

Justification: ....

....

(d) Let $a \in \{0,1\}^6$, $a \neq 0$, and let $r \in_{\text{u.a.r.}} \{0,1\}^6$.
Then $\Pr\left[r^{T}a = 0 \mod 2\right]$ equals exactly $\frac{1}{2}$.

[ ] False [ ] True

Justification: ....

....

Random(ized) Search Trees

Exercise 2.1 (10 points)

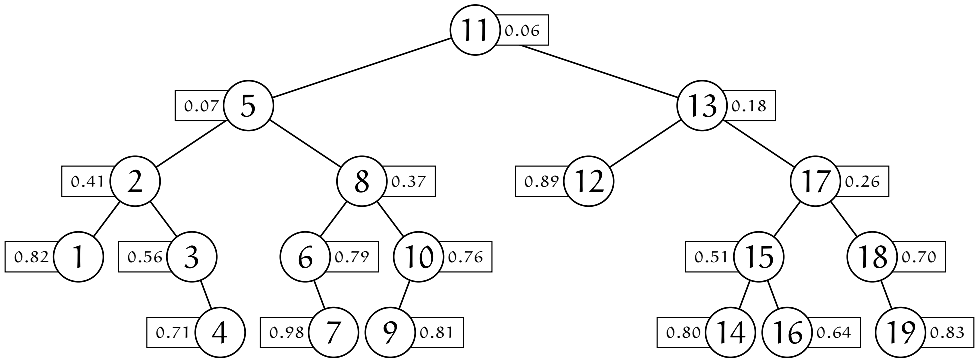
Let R_n denote the number of nodes with empty right subtree in a random search tree on n keys ($n \in \mathbf{N}_0$). Find a recurrence for $r_n := \mathbf{E}[R_n]$ and solve it.

Exercise 2.2 (5 points)

Let $x \in [n]$, and let B denote the indicator variable for the event that *node* x *is a leaf or node* x *is the root* in a random search tree on n keys. Express B in terms of the indicator variables $A_i^j = [\,\text{node } j \text{ is ancestor of node } i\,]$, using only arithmetic operations.

Exercise 2.3 (5 points)

The node 13 has been inserted into a treap, and the resulting treap is shown below. How many rotations were performed during the insertion of node 13?

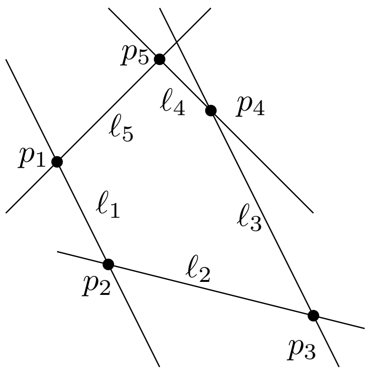


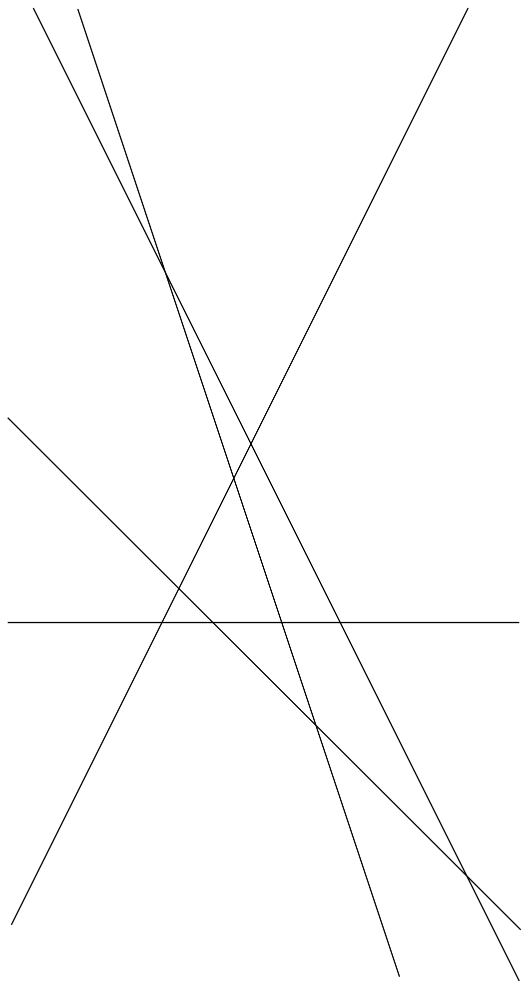
Point Location

Exercise 3.1 (5 points)

The picture on the right consists of the duals p_i^* of the points p_i on the left. Both pictures were drawn in a standard coordinate system, with the y-axis pointing up (and then they were scaled to fit nicely on the page).

Identify the points $\ell_1^*, \ldots, \ell_5^*$ and label each of them clearly. You do not need to justify your answer.





Exercise 3.2 (10 points)

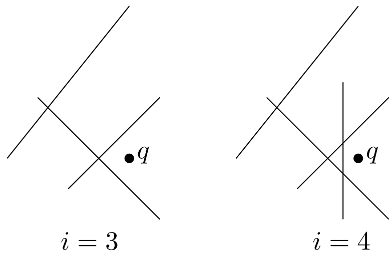
Let $q \in \mathbf{R}^2$ be a point, and let L be a set of $n \geq 5$ lines in $\mathbf{R}^2$. We make the following assumption: For every circle $C \subset \mathbf{R}^2$, at most five lines from L are tangent to C.

Let $\ell_1, \ldots, \ell_n$ be a permutation of L chosen uniformly at random, and suppose that $\ell_1, \ldots, \ell_n$ are inserted one by one into an initially empty line arrangement. For every $i \in [n]$ we consider "the line arrangement at time i",

$$A_i = \{\ell_1, \ldots, \ell_i\},$$

and we define $A'_i \subseteq A_i$ as the subset that consists of those lines whose distance to q is minimal. Let $N = \left|\bigcup_{i \in [n]} A'_i\right|$ denote the number of distinct lines nearest to q encountered in this process.

In the example on the side, we have $|A'_3| = 2$ and $|A'_4| = 1$.



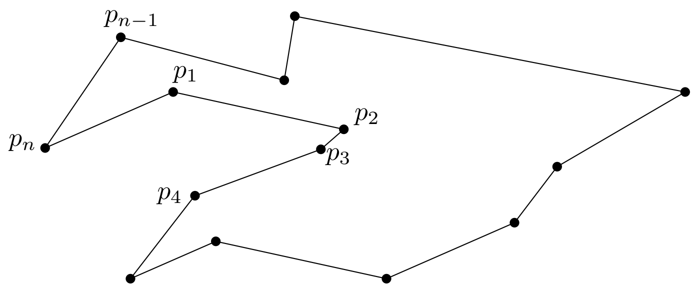
Show that we have $\mathbf{E}[N] \leq 5 \ln n$.

Exercise 3.3 (5 points)

You are given a list of points $p_1 \ldots, p_n \in \mathbb{R}^2$ with the guarantee that they are the vertices of a polygon in counter-clockwise order. The polygon is not necessarily convex, but without holes, as in the picture. Explain how to preprocess the list so that you can determine for any query point $q \in \mathbb{R}^2$ in *less than* linear expected time whether it lies inside, outside, or on the boundary of the polygon. Explain also

(i) how you answer queries and how long this takes, and

(ii) how much space your data structure uses.

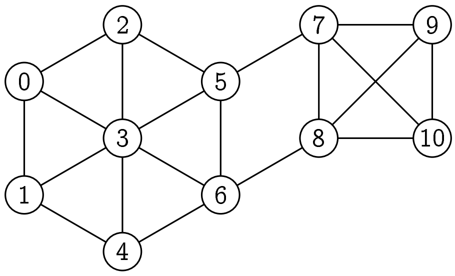


Minimum Cut

Exercise 4.1 (5 points)

Let G be the (multi)graph with vertices $0, \dots, 10$ depicted below, and let e be an edge of G chosen uniformly at random. Calculate the value of $\Pr[\mu(G) = \mu(G/e)]$ and $\mathbf{E}[\mu(G/e)]$.

To make your life easier in this exercise, you may make statements of the type "the minimum cuts in the graph are …" without proof.



Exercise 4.2 (10 points)

We have seen in the lecture that the success probability for a single run of MinCut on any n-vertex input graph satisfies $P(n) \geq 1/(1 + \log_c n)$ for all $n \geq 2$, where $c = 1.2$.

Describe how to compute the size of a minimum cut with a success probability of

$$P'(n) \geq 1 - \frac{1}{n^n}.$$

Calculate explicitly (without O-notation or similar) how often your algorithm should call MinCut in order to satisfy this bound, and then show that the overall running time is $O(n^3 \log^3 n)$.