ETH

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Algorithms, Probability, and Computing Exercises KW39 HS25

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A: Wed 14–16 CAB G 56

  Group B: Wed 14–16 CAB G 57

  Group C: Wed 16–18 CAB G 56

  Group D: Wed 16–18 CAB G 57

- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

The following exercises will be discussed in the exercise classes on September 24, 2025. Please hand in your solutions via Moodle, no later than 2 pm at September 23.

Exercise 1

Let $G = (V, E)$ be a connected graph with weights $w : E \to \mathbb{R}$ on the edges, and define the edge boundary of set S to be

$$\partial(S) := \{\{u, v\} \in E : u \in S, v \in V \setminus S\}.$$

Assume that for every non-empty vertex set $S \subset V$, the edge with the minimum weight in $\partial(S)$ is unique.

Prove that G has a unique MST. Conclude that if the weight function w is injective (i.e., no two edges have the same weight), G contains exactly one MST.

Exercise 2

You already know that for a connected graph $G = (V, E)$, with $n = |V|$ and $m = |E|$, the expected running time of *Randomized Minimum Spanning Tree Algorithm (G)* (see page 5 in the lecture notes) is equal to $\mathcal{O}(m)$.

(i) Prove that the worst-case running time of the algorithm is equal to $\mathcal{O}(\min\{n^2, m \log n\})$.

(ii) Prove that the running time of the algorithm is equal to $\mathcal{O}(m)$ with probability $1 - o(1)$ in the following two steps.

(a) Let $D(n, m)$ be the worst-case running time of the recursive algorithm without considering the two recursive calls, and let $T(n, m)$ be the worst-case running time of the recursive algorithm. It is clear that $\mathbf{D(n, m) = \mathcal{O}(n + m)}$, and by (i), $T(n, m) = \mathcal{O}(\min\{n^2, m \log n\})$. Figure 1 represents a binary tree of running times in which every first recursive call works on a graph with at most $\frac{n}{8}$ vertices and at most $\frac{3}{4}m$ edges, every second recursive call works on a graph with at most $\frac{n}{8}$ vertices and at most $\frac{3}{8}n$ edges, and the worst-case function $T(n^{2/5}, \infty)$ will be applied when the number of vertices has shrunk down to $n^{2/5}$.

Prove that there exists a constant $c_3 > 0$ such that the sum of all running times in the nodes of the tree depicted in Figure 1 is bounded from above by $c_3 \cdot (n + m)$.

(b) Prove that for any connected input graph G, the running time of the algorithm is dominated by the sum over all nodes in the tree depicted in Figure 1 with probability $1 - o(1)$ (i.e., a number that tends to 1 as $n \to \infty$).

Hint: Let G_1 and G_2 be the two graphs for the first and second recursive calls, respectively. Call G_1 bad if G_1 has more than $\frac{n}{8}$ vertices or more than $\frac{3}{4}m$ edges, and call G_2 bad if G_2 has more than $\frac{n}{8}$ vertices or $\frac{3}{8}n$ edges. You might apply the Chernoff bound to bound the probability that G_1 or G_2 is bad. For the Chernoff bound, see the help sheet on the website of the course.

Exercise 3

For a graph $G = (V, E)$, a cut is the partition of the vertex set V into two disjoint sets V_1 and V_2 and the size of the cut is the number of edges between V_1 and V_2.

(i) Assume that $|V|$ is even. We say a cut is balanced if $|V_1| = |V_2| = |V|/2$. Prove that there always exists a balanced cut of size at least $|E|/2$.

Hint: Since you want to prove the existence, you might apply randomness.

(ii) Define $d_S(v)$ to be the number of neighbors of vertex v in a set $S \subseteq V$, i.e. $d_S(v) := |\{u \in S : \{v, u\} \in E\}|$. Now, consider the following algorithm. Partition the vertex set V into two arbitrary sets V_1 and V_2. As far as there is a vertex $v \in V_1$ (or $v \in V_2$) for which $d_{V_1}(v) > d_{V_2}(v)$ (resp. $d_{V_2}(v) > d_{V_1}(v)$) move v from V_1 to V_2 (resp. from V_2 to V_1). Prove that this algorithm terminates and generates a cut of size at least $|E|/2$.

$$D(n, m)$$

$$D(\tfrac{1}{8}n, \tfrac{3}{4}m) \qquad\qquad D(\tfrac{1}{8}n, \tfrac{3}{8}n)$$

$$D(\tfrac{1}{8^2}n, \tfrac{3}{4}(\tfrac{3}{4}m)) \quad D(\tfrac{1}{8^2}n, \tfrac{3}{8^2}n) \qquad D(\tfrac{1}{8^2}n, \tfrac{3}{4}(\tfrac{3}{8}n)) \quad D(\tfrac{1}{8^2}n, \tfrac{3}{8^2}n)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$T(n^{2/5}, \infty)\ T(n^{2/5}, \infty)\ T(n^{2/5}, \infty)\ T(n^{2/5}, \infty) \qquad T(n^{2/5}, \infty)\ T(n^{2/5}, \infty)\ T(n^{2/5}, \infty)\ T(n^{2/5}, \infty)$$
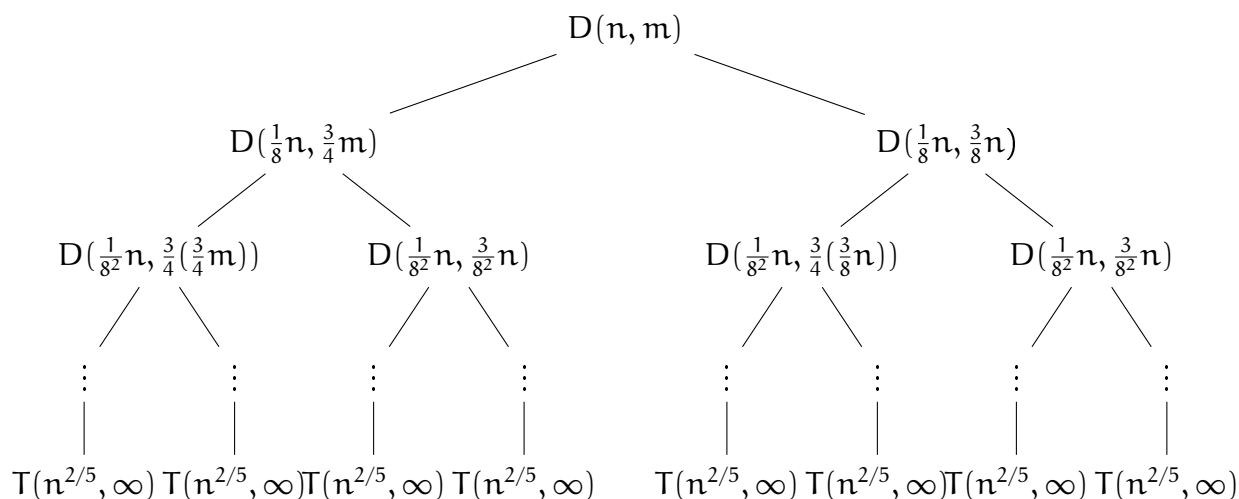
Figure 1: A binary tree of very specific running times that we use in order to prove an upper bound on the actually observed running time of the algorithm.

Exercise 4

In a city there are n houses $h_1, \cdots, h_n$, each of which is in need of a water supply. It costs c_i to build a well at house h_i, and it costs w_{ij} to build a pipe in between houses h_i and h_j. A house can receive water if either there is a well built there or there is some path of pipes to a house with a well. Give an algorithm to find the minimum cost to supply every house with water.

Hint: There is a short solution.