ETH

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Rasmus Kyng, Bernd Gärtner, David Steurer, Vera Traub

Algorithms, Probability, and Computing **Exercises KW40** **HS25**

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A: Wed 14–16 CAB G 56

  Group B: Wed 14–16 CAB G 57

  Group C: Wed 16–18 CAB G 56

  Group D: Wed 16–18 CAB G 57

- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

The following exercises will be discussed in the exercise classes on October 1, 2025. Please hand in your solutions via Moodle, no later than 2 pm at September 30.

Exercise 1

You recall that the algorithm BasicMinCut computes a guess for the size of a minimum cut of a (multi)graph G by repeatedly contracting a uniformly random edge until there are only two vertices left and then returning the number of edges running between these two vertices.

As usual, denote the size of a minimum cut of G by $\mu(G)$. We have derived in the lecture that the number L_G which BasicMinCut outputs (on input G) is at least $\mu(G)$, and $\Pr[L_G = \mu(G)] = \Omega(n^{-2})$.

Consider the following slightly modified algorithm BasicMinCut': just like BasicMinCut, it repeatedly contracts a uniformly random edge until there are only two vertices left. But instead of just returning the number of edges between those two vertices in the very end, it returns the smallest degree of any vertex observed during the execution of the algorithm. That is if $G_0, G_1, G_2, \ldots, G_{n-2}$ is the sequence of graphs encountered, with $G_0 = G$ and $|V(G_{n-2})| = 2$, it returns

$$L_G := \min_{0 \le i \le n-2} \min_{v \in V(G_i)} \deg(v).$$

Prove that

(a) BASICMINCUT' can be implemented so as to run in time $\mathcal{O}(n^2)$,

(b) $L_G \geq \mu(G)$ always holds,

(c) for any fixed $\alpha > 0$, the sucess probability

$$p_\alpha(n) := \min_{G \text{ a graph on } n \text{ vertices}} \Pr\left[L_G \leq (1+\alpha)\mu(G)\right]$$

satisfies the recurrence

$$p_\alpha(n) \geq \left(1 - \frac{2}{(1+\alpha)n}\right) p_\alpha(n-1).$$

Using (c), one can prove that for any fixed $\alpha > 0$, $p_\alpha(n) \in \Omega(n^{\frac{-2}{1+\alpha}})$, but this is just calculation and we do not ask you to do this here.

Exercise 2

Prove that a connected (multi)graph G on n vertices cannot have more than $\binom{n}{2}$ minimum cuts.

Exercise 3

Let (V, E) be an undirected graph. For every subset of vertices S we associate the cut $C(S) := E(S, V \setminus S)$. Define a function $f : 2^V \to \mathbb{R}$ by $f(S) := |C(S)|$ for $S \subseteq V$. In other words $f(S)$ is the size of the cut $C(S)$.

(a) Show that the function f is *sub-modular*, meaning that

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B), \qquad \forall A, B \subseteq V.$$

(b) Let $A, B \subset V$ be sets so that $C(A)$ and $C(B)$ are minimum cuts[1] and suppose that $A \cap B \neq \emptyset$ and $A \cup B \neq V$. Prove that in this case, also $C(A \cap B)$ and $C(A \cup B)$ must be minimum cuts of G.

(c) For two vertices $s, t \in V$ and a nonempty pure subset $S' \subset V$, $C(S')$ is a s-t-cut if $s \in S'$ and $t \in V \setminus S'$. A s-t-cut $C(S')$ is called *minimum* if $|C(S')| \leq |C(S'')|$ for all s-t-cuts $C(S'')$. Let $S \subset V$ be such that $C(S)$ is a minimum s-t-cut, $s \in S$ and $|S|$ is as small as possible with the first two properties. Show that S is unique.

[1] For a *nonempty pure* subset S of V, $C(S)$ is a minimum cut if $f(S) = \min_{S' \subset V, S' \neq \emptyset} f(S')$

Exercise 4

Let G be a connected multigraph with $n \geq 3$ vertices and that has a minimum cut of size $k = \mu(G)$. The probabilities $\Pr[\mu(G) \neq \mu(G/e)]$ below are always with respect to a uniformly at random chosen edge e from the graph G.

(a) Assume $k = 1$. Prove that $\Pr[\mu(G) \neq \mu(G/e)] \leq \frac{1}{n}$.

(b) Assume $k = 1$. Show that the basic random edge contraction algorithm BasicMinCut(G) outputs the correct number $\mu(G) = 1$ with probability at least $\frac{2}{n}$.

(c) For any k show that if G has at least 3 vertices with degree k, then $\Pr[\mu(G) \neq \mu(G/e)] = 0$.

(d) For any k, show that $\Pr[\mu(G) \neq \mu(G/e)] \leq \frac{2k}{n(k+1)-t}$ where t is a constant. Prove that the bound holds for some $t \leq 1$.