ETH

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Algorithms, Probability, and Computing — Exercises KW42 — HS25

General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  Group A: Wed 14–16 CAB G 56

  Group B: Wed 14–16 CAB G 57

  Group C: Wed 16–18 CAB G 56

  Group D: Wed 16–18 CAB G 57

- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

The following exercises will be discussed in the exercise classes on October 15, 2024. Please hand in your solutions via Moodle, no later than 2 pm at October 14.

Exercise 1

For a permutation π on the keys $\{1..n\}$, let T_π be the search tree that we obtain from inserting all n keys, one after the other, in the order given by π.

Prove: If π is drawn uniformly at random, then T_π has the same distribution as $\tilde{\mathcal{B}}_{[n]}$ from the lecture.

Exercise 2

Suppose you are given a finite set $S \subseteq \mathbf{R}$, $2 \leq |S|$, which is to be preprocessed so that for query $q \in \mathbf{R}$ the answer is 'the' set $\{b_1, b_2\} \subseteq S$ of the two closest numbers in S (i.e. $\max\{|b_1 - q|, |b_2 - q|\} \leq \min_{a \in S \setminus \{b_1, b_2\}} |a - q|$). Follow the locus approach for the problem and describe the resulting partition of regions of equal answers (and be aware of the ambiguity issue, i.e. the 'the' has to be taken with caution).

Exercise 3

Given a sorted sequence $a_0 < a_1 < \cdots < a_{n-1}$ of n real numbers, we consider the convex polygon C with vertices $\left((a_i, a_i^2)\right)_{i=0}^{n-1}$. For $k \in \mathbf{R}$, show that the line with equation $y = 2kx - k^2$ intersects C iff $k \in \{a_0, a_1, \ldots, a_{n-1}\}$.

REMARK: This exercise is supposed to exhibit that deciding whether a line intersects a convex polygon cannot be easier than deciding whether a query key is in a given set of keys.

Exercise 4

Recall the algorithm for counting the points below a query line: We interpreted the question in the dual setting, so we were looking for the number of lines above a query point. Prove the space bound in lemma 3.6, which was left as an exercise:

$$\sum_{\nu \text{ inner node}} |\bar{S}_\nu| \;\leq\; 2n^2$$

where, as you may recall,

- n is the number of lines,
- ν ranges over the nodes of the data structure used in the algorithm, which has one node for every level of the line arrangement;
- S_ν is the set of x-coordinates of the corresponding level;
- $\bar{S}_\nu$ is the 'enhanced' set: If a node ν has no child which is an inner node, $\bar{S}_\nu = S_\nu$. Otherwise, $\bar{S}_\nu$ is obtained from S_ν by adding every other value from each of the sets $\bar{S}_u$, u a non-leaf child of ν.