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**Algorithms, Probability, and Computing** **Exercises KW47** **HS25**

## General rules for solving exercises

- When handing in your solutions, please write your exercise group on the front sheet:

  **Group A**: Wed 14–16 CAB G 56

  **Group B**: Wed 14–16 CAB G 57

  **Group C**: Wed 16–18 CAB G 56

  **Group D**: Wed 16–18 CAB G 57

- This is a theory course, which means: if an exercise does not explicitly say "you do not need to prove your answer", then a formal proof is **always** required.

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The following exercises will be discussed in the exercise classes on November 19, 2024. Please hand in your solutions via Moodle, no later than 2 pm at November 18.

## Exercise 1

Show that every feasible point of the Tight Spanning Tree LP is feasible in the Loose Spanning Tree LP – without using theorem 4.11.

## Exercise 2

Consider the following linear program, almost the Tight Spanning Tree LP, it seems:

**Some LP** for graph $G = (V, E), c \in \mathbf{R}^E$

$$\begin{array}{llrcl} \min & c^T x & & & \\ \text{subject to} & & \sum_{e \in E} x_e & = & n \\ & & \sum_{e \in E \cap \binom{S}{2}} x_e & \leq & |S| - 1, \quad \text{for all } S \subseteq V,\ \emptyset \neq S \neq V, \text{ and} \\ & & 1 \geq x_e & \geq & 0, \quad \text{for all } e \in E. \end{array}$$

What are the edge sets corresponding to vectors $x \in \{0,1\}^E$ feasible in Some LP?

## Exercise 3

Let $D = (V, A)$ be a directed graph and let $s, t \in V$. To any vertex set $S \subseteq V$ we associate a *cut* $C(S) \subseteq A$ that consists of all arcs between $S$ and $V \setminus S$. We say that $C(S)$ is an *s-t cut* if $s \in S$ and $t \notin S$. We say that $C(S)$ is a *strong* s-t *cut* if it is an s-t cut and if all edges in $C(S)$ are directed away from $V \setminus S$. See Figure ?? for an example.

In this exercise we will prove the following lemma and see that it is a special case of the Farkas lemma we have seen in the lecture. Informally, it says that there is a simple certificate for both proving and disproving the existence of a directed s-t path in $D$.

**Lemma 1** (Farkas lemma for s-t-paths). *Exactly one of the following two statements holds for any directed graph $D = (V, A)$ and for any two vertices $s, t \in V$.*

i) *There exists a directed s-t path.*

ii) *There exists a strong s-t cut.*

For every vertex $v \in V$ let $\delta(v)^+ \subseteq A$ denote the arcs that are outgoing from $v$ and let $\delta(v)^- \subseteq A$ denote the arcs that are incoming to $v$.

(a) Show that there is a directed s-t path in $D$ if and only if the following system of equations and inequalities has a solution over the real valued variables $\{x_e \mid e \in A\}$.

$$\begin{aligned} \forall v \in V: &\quad \sum_{e \in \delta(v)^+} x_e - \sum_{e \in \delta(v)^-} x_e = \begin{cases} 0 & \text{if } v \in V \setminus \{s, t\} \\ 1 & \text{if } v = s \\ -1 & \text{if } v = t \end{cases} \\ \forall e \in A: &\quad x_e \geq 0 \end{aligned}$$

(b) Prove Lemma ?? by applying some version of Farkas lemma to the system in (a).

(c) Prove Lemma ?? directly without using (a) or Farkas lemma.

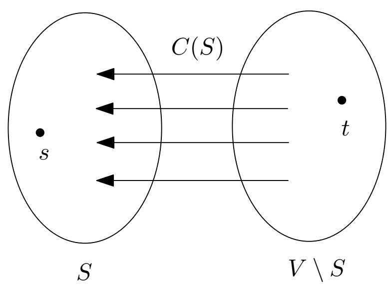


Figure 1: An illustrative example of a strong s-t cut. The cut $C(S)$ is a strong s-t cut because all edges in $C(S)$ are directed away from $V \setminus S$.

## Exercise 4

Suppose we are running the checking algorithm for matrices over $GF(2)$, i.e. numbers are $\{0, 1\}$ with addition and multiplication $\mod 2$. Show that in one iteration the success probability of detecting an error in the supposed product matrix C is exactly $\frac{1}{2}$, in case matrix C is wrong in exactly one row.

## Exercise 5

For $n \in \mathbf{N}$, let $A \in \mathbf{R}^{n \times n}$ be a non-zero matrix (i.e. not all entries are 0) and let $x$ be a vector u.a.r. from $\{-1, 0, +1\}^n$. Show that the probability that the vector $Ax$ is non-zero is at least $2/3$.

## Exercise 6

Given a finite set $S$ of rational numbers and positive integers $d$ and $n$, $d \leq |S|$, find a polynomial $p(x_1, x_2, \ldots, x_n)$ of degree $d$ for which the Schwartz–Zippel theorem is tight. That is, the number of $n$-tuples $(r_1, \ldots, r_n) \in S^n$ with $p(r_1, \ldots, r_n) = 0$ is $d|S|^{n-1}$.